

HEREDITARY C*-SUBALGEBRA LATTICES

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ABSTRACT. We investigate the connections between order and algebra in the hereditary C*-subalgebra lattice $\mathcal{H}(A)$ and *-annihilator ortholattice $\mathcal{P}(A)^\perp$. In particular, we characterize \vee -distributive elements of $\mathcal{H}(A)$ as ideals, answering a 25 year old question, allowing the quantale structure of $\mathcal{H}(A)$ to be completely determined from its lattice structure. We also show that $\mathcal{P}(A)^\perp$ is separative, allowing for C*-algebra type decompositions which are completely consistent with the original von Neumann algebra type decompositions.

1. INTRODUCTION

1.1. Motivation. Hereditary C*-subalgebras $\mathcal{H}(A)$ of a C*-algebra A have long been considered analogs of open sets. Given the fundamental role open subsets and their lattice structure play in topological spaces (as more clearly seen in the point-free topology of frames and locales), one would expect us by now to have a deep understanding of $\mathcal{H}(A)$, with numerous theorems relating algebraic properties of A to order properties of $\mathcal{H}(A)$. But on the contrary, our knowledge of $\mathcal{H}(A)$ is still quite limited, and the study of $\mathcal{H}(A)$ has remained very much on the periphery of mainstream C*-algebra research. Needless to say, we see this as a somewhat strange state of affairs.

Another perplexing trend in operator algebras is the early divergence of von Neumann algebra and C*-algebra theory. Again, one would naturally expect that, as von Neumann algebras form a nice subclass of C*-algebras, much inspiration could be drawn from looking at the von Neumann algebra theory and trying to generalize it in various ways to C*-algebras. But yet again, we rarely see this happening, particularly in modern C*-algebra research, with much of the von Neumann algebra theory dismissed long ago as either inapplicable or irrelevant to general C*-algebras.

In fact, this is no coincidence, as it is precisely this more topological, order theoretic approach that is required to generalize some of the basic von Neumann algebra theory. This can be seen in [Bic14a] and [Bic14b], and we continue in this direction in the present paper, using mainly classical theory to prove a number of new, fundamental and very general C*-algebra results regarding $\mathcal{H}(A)$, and its subset of *-annihilators $\mathcal{P}(A)^\perp$. We hope this might spur on further research in this largely neglected subfield of C*-algebra theory.

1.2. Outline. We give the necessary basic definitions and assumptions for the rest of the paper in §2, which the reader is welcome to skim over and refer back to only when unfamiliar terminology or notation appears later on. We start the paper proper with a brief note on compactness in §3. Following that, we examine the

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semicomplement structure of $\mathcal{H}(A)$ in §4 and obtain various characterizations of strong orthogonality in Theorem 1. Next, in §5, we exhibit a correspondence, in the unital case, between two of the most common objects that appear in lattices and C^* -algebras, namely complements and projections. We then show in §6 that the superficially similar notion of a \wedge -pseudocomplement turns out to have a quite different algebraic characterization in $\mathcal{H}(A)$, namely as an annihilator ideal. Next, in §7, we show that arbitrary ideals in $\mathcal{H}(A)$ can be characterized as the \vee -distributive elements, answering a long-standing question from [BRVdB89]. Quantales, as introduced in [Mul86], have often been considered the appropriate non-commutative analogs of locales, and this characterization shows that the natural quantale structure on $\mathcal{H}(A)$ is, in fact, completely determined by its lattice structure.

Another result of fundamental importance is the fact that $*$ -annihilators satisfy a strong version of the SSC property, which we show in §8 (although, as we show in Example 4, this property does not quite characterize the $*$ -annihilators). We then examine the $*$ -annihilators $\mathcal{P}(A)^\perp$ as an ortholattice in its own right in §9. It turns out that many of the order theoretic concepts examined in $\mathcal{H}(A)$ converge in the case of $\mathcal{P}(A)^\perp$, coinciding with the annihilator ideals, as we show in Theorem 9. Lastly, we outline in §10 how the separativity/SSC property of $\mathcal{P}(A)^\perp$ allows for order theoretic type decompositions that are completely consistent with the original von Neumann algebra type decompositions.

2. PRELIMINARIES

2.1. Posets. Recall that a *poset* is simply a partially ordered set, i.e. a set \mathbb{P} together with a binary relation \leq on \mathbb{P} that is transitive ($p \leq q \leq r \Rightarrow p \leq r$), reflexive ($p \leq p$) and anti-symmetric ($p \leq q \leq p \Rightarrow p = q$). A *subposet* of \mathbb{P} is simply a subset of \mathbb{P} under the same ordering. For any poset \mathbb{P} there is a dual poset \mathbb{P}^* such that the underlying set is the same but the order reversed. Consequently, all poset definitions and results also have duals.

Infimums (greatest lower bounds) are, when they exist, denoted by $\bigwedge S$ (in particular, $\bigwedge \emptyset$ is the maximum of \mathbb{P} , when it exists). We also write $p \wedge q$ for $\bigwedge \{p, q\}$. A \wedge -lattice (or *meet semilattice*) is a poset \mathbb{P} where $p \wedge q$ exists, for all $p, q \in \mathbb{P}$. A \wedge -sublattice of a \wedge -lattice \mathbb{P} is a subset S closed under \wedge . We similarly define \wedge -(sub)lattices. Dually, we denote *supremums* (least upper bounds) by \bigvee and \vee and define \vee -(sub)lattices and \bigvee -(sub)lattices accordingly. In fact, \mathbb{P} is a \wedge -lattice precisely when \mathbb{P} is a \bigvee -lattice, in which case \mathbb{P} is called a *complete* lattice. For the remainder of this paper, let us assume that

$$\mathbb{P} \text{ is a poset with } 0 = \bigwedge \mathbb{P} \text{ and } 1 = \bigvee \mathbb{P}.$$

Given a set X , we always order its subsets $\mathcal{P}(X)$ by \subseteq (inclusion), which makes $\mathcal{P}(X)$ a complete lattice in which \bigvee is \bigcup (union) and \bigwedge is \bigcap (intersection). Many of the posets we consider in this paper are subposets of $\mathcal{P}(X)$, for some X , like the open subsets $\mathcal{P}(X)^\circ$ of a topological space X . Here $\mathcal{P}(X)^\circ$ is, by definition, nothing more than some \bigvee -sublattice and \wedge -sublattice of $\mathcal{P}(X)$ containing $\{\emptyset, X\}$ (in fact, as $\mathcal{P}(X)^\circ$ is a \bigvee -sublattice, we automatically have $\emptyset = \bigvee \emptyset \in \mathcal{P}(X)^\circ$).

For another poset central to this paper, consider a C^* -algebra A , i.e. a Banach $*$ -algebra with $\|a^*a\| = \|a\|^2$, for all $a \in A$. The *positive* elements A_+ of A are precisely those of the form a^*a , for some $a \in A$. We order A_+ in the usual way by

$$a \leq b \iff b - a \in A_+.$$

As in [Ped79] §1.5, we call a C*-subalgebra B of A *hereditary* when, for all $a, b \in A_+$,

$$(2.1) \quad a \leq b \in B \quad \Rightarrow \quad a \in B,$$

although they could equivalently be defined as closed self-adjoint bi-ideals (see [Bla13] Proposition II.3.4.2 and Corollary II.5.2.9) or *-annihilators with respect to the canonical action of A on A^* (see [Eff63] Theorem 2.5). These hereditary C*-subalgebras $\mathcal{H}(A)$ form a complete lattice¹ in which \bigwedge is \bigcap .

2.2. Commutative C*-Algebras. From now on, we assume that

X is a topological space.

We denote the continuous and bounded continuous functions from X to \mathbb{C} by $C(X)$ and $C^b(X)$ respectively. With pointwise operations and the supremum norm, $C^b(X)$ becomes a commutative C*-algebra. For any $Y \subseteq X$, we have a hereditary C*-subalgebra of $C^b(X)$ given by

$$B_Y = \{f \in C^b(X) : f[X \setminus Y] = \{0\}\}.$$

On the other hand, for any $B \subseteq C^b(X)$, we have an open subset of X given by

$$O_B = \bigcup_{f \in B} f^{-1}[\mathbb{C} \setminus \{0\}].$$

When X is completely regular, hereditary C*-subalgebras of $C^b(X)$ distinguish open subsets of X in the sense that $O = O_{B_O}$, for all $O \in \mathcal{P}(X)^\circ$. When X is compact $C^b(X) = C(X)$ and open subsets of X distinguish hereditary C*-subalgebras of $C(X)$ in the sense that $B = B_{O_B}$, for all $B \in \mathcal{H}(C(X))$. More generally, this holds when X is locally compact and we consider the C*-subalgebra $C_0(X)$ of functions in $C(X)$ that vanish at infinity, i.e. those $f \in C(X)$ such that $f^{-1}[\mathbb{C} \setminus \mathbb{C}^\epsilon]$ is compact, for all $\epsilon > 0$, where $\mathbb{C}^\epsilon = \{\lambda \in \mathbb{C} : |\lambda| < \epsilon\}$.

The Gelfand representation theorem tells us that these are, up to isomorphism, the only commutative C*-algebras. More specifically, every commutative C*-algebra A is isomorphic to $C_0(X)$ for some locally compact Hausdorff space X . So in this case, by the previous paragraph, we get a natural bijection between open subsets of X and hereditary C*-subalgebras of A . This is why hereditary C*-subalgebras of even non-commutative C*-algebras A are considered to be analogs of open subsets.²

¹which is isomorphic to many other lattices defined from A , e.g. the lattice of closed left or right ideals in A , closed cones in A_+ (see [Eff63] Theorem 2.4), norm filters in (unital) A_+^1 (see [Bic13b] Corollary 3.4), weak* closed faces of A_+^{*1} (see [Ped79] Proposition 3.11.9), or open projections in A'' (see [Ake69] Proposition II.2). Indeed, we will often work with open projections, as in [Ake69], [Ake70] and [Ake71], but $\mathcal{H}(A)$ has the advantage that each $B \in \mathcal{H}(A)$ remains in the category of C*-algebras, so concepts like commutativity naturally carry over.

²By this criterion, there are also other subsets of A that could be considered as open subset analogs. For example, we could consider closed ideals, which are precisely the hereditary C*-subalgebras when A is commutative. However, the closed ideal structure of A can yield little information about (e.g. simple) non-commutative A . Alternatively, we could consider more general closed bi-ideals ($B \subseteq A$ with $\overline{BAB} \subseteq B$) which, again, are precisely the hereditary C*-subalgebras when A is commutative. But this would take us outside the category of C*-algebras and into the realm of non-self-adjoint operator algebras.

2.3. General C*-Algebras. Another important class of C*-algebras consists of operators on a Hilbert space. Indeed, by the GNS construction (see [Ped79] §3.3), every C*-algebra is isomorphic to a C*-subalgebra A of $\mathcal{B}(H)$, the C*-algebra of all bounded linear operators on a Hilbert space H . In this case, following standard practice, we denote the commutant of A in $\mathcal{B}(H)$ by

$$(2.2) \quad A' = \{b \in \mathcal{B}(H) : \forall a \in A (ab = ba)\},$$

and recall that von Neumann's double commutant theorem (see [Ped79] Theorem 2.2.2) says that A'' coincides with the weak (and strong) closure of A in $\mathcal{B}(H)$. We also define the multiplier algebra (see [Ped79] §3.12) $\mathcal{M}(A)$ of A by

$$\mathcal{M}(A) = \{a \in \mathcal{B}(H) : aA, Aa \subseteq A\}.$$

Also important are the *projections*

$$\mathcal{P}(A) = \{p \in A : p^2 = p = p^*\}.$$

We also define an operation $^\perp$ on projections in $\mathcal{B}(H)$ by $p^\perp = 1 - p$ and note that

$$p \leq q \quad \Leftrightarrow \quad pq^\perp = 0.$$

For $a \in A$, let A_a denote the hereditary C*-subalgebra of A generated by a , i.e.

$$A_a = \overline{a^*Aa} \vee \overline{aAa^*}.$$

More generally, for $a \in A''$, let $A_a = A''_a \cap A$, so if $p \in \mathcal{P}(A'')$ then $A_p = pAp \cap A$. On the other hand, if $B \in \mathcal{H}(A)$ then we define $p_B = \bigvee B^\perp_+ \in \mathcal{P}(A'')$, where $A^\lambda = \{a \in A : \|a\| \leq \lambda\}$ denotes the closed λ -ball about 0. Note that B^\perp_+ is generally not a lattice however, as B is a C*-algebra, B has an increasing approximate unit (see [Ped79] Theorem 1.4.2) which has a supremum $\bigvee B^\perp_+$ in A''_+ . We call such projections *open* (see [Ped79] Proposition 3.11.9 for some equivalent definitions), denoting them by $\mathcal{P}(A'')^\circ = \{p_B : B \in \mathcal{H}(A)\}$. For $p \in \mathcal{B}(H)$, we naturally define the *interior* p° of p to be the largest open projection below p , i.e. $p^\circ = p_{A_p}$. Note that suprema in $\mathcal{P}(A'')^\circ$ agree with suprema in $\mathcal{P}(A'')$ (see [Ake69] Proposition II.5³), but the same can not be said for infima (see [Ake69] Example II.6). We also define the *closure* \overline{p} of p by $\overline{p} = p^{\perp\circ\perp}$ and call p *closed* when $p = \overline{p}$, i.e. when p^\perp is open. The closed projections will be denoted by $\overline{\mathcal{P}(A'')}$ which, as a poset, is the dual of $\mathcal{P}(A'')^\circ$.

2.4. Ideals. Let $\mathcal{I}(A)$ denote the closed ideals of A . We have $\mathcal{I}(A) \subseteq \mathcal{H}(A)$ (see [Ped79] Theorem 1.5.2 and Corollary 1.5.3) and the corresponding open projections are precisely those in A' (see [Ped79] 3.11.10). In fact, p° (and \overline{p}) lies in A' whenever $p \in \mathcal{P}(A'' \cap A')$, i.e.

$$\mathcal{P}(A'' \cap A')^\circ = \mathcal{P}(A'')^\circ \cap A' = \{p_I : I \in \mathcal{I}(A)\}.$$

To see this, take $I \in \mathcal{I}(A)$ and note that I'' is then a weakly closed ideal in A'' with unit p_I . Thus, for any $a \in A$, we have $ap_I, p_Ia \in I''$ and hence $ap_I = p_Iap_I = p_Ia$, i.e. $p_I \in A'$. While if $p \in \mathcal{P}(A'' \cap A')$ then $p^\perp \in A'$ so, for any $a, b \in A$ with $ap^\perp = 0$, we have $abp^\perp = ap^\perp b = 0$ and $bap^\perp = 0$, i.e. $A_p = \{a \in A : ap^\perp = 0\} \in \mathcal{I}(A)$ and hence $p^\circ = p_{A_p} \in \{p_I : I \in \mathcal{I}(A)\}$.

³Throughout [Ake69], A is assumed to be unital and it is the universal representation of A that is considered. However, these assumptions are not necessary for this particular result. Indeed, given $p, q \in \mathcal{P}(A'')^\circ$, we certainly have $p, q \leq (p \vee q)^\circ$ and hence $p \vee q \leq (p \vee q)^\circ \leq p \vee q$.

For any $a \in A_+''$, we define the *central cover* $c(a)$ as in [Ped79] 2.6.2, specifically $c(a) = \bigwedge \{a' \in (A'' \cap A')_+ : a \leq a'\}$. Likewise, any $B \in \mathcal{H}(A)$ has an *ideal cover* $\overline{\text{span}}(ABA) = \bigcap \{I \in \mathcal{I}(A) : B \subseteq I\}$. In fact, these covers correspond in the sense that, for any $B \in \mathcal{H}(A)$, we have

$$c(p_B) = p_{\overline{\text{span}}(ABA)}.$$

For $p_{\overline{\text{span}}(ABA)} \in A'' \cap A'$ and $p_B \leq p_{\overline{\text{span}}(ABA)}$ so $c(p_B) \leq p_{\overline{\text{span}}(ABA)}$. But also $A_{c(p_B)} \in \mathcal{I}(A)$ and $B \subseteq A_{c(p_B)}$ so $\overline{\text{span}}(ABA) \subseteq A_{c(p_B)}$ hence $p_{\overline{\text{span}}(ABA)} \leq c(p_B)$.

2.5. The Reduced Atomic Representation. From now on we assume that

A is a C*-algebra identified with its reduced atomic representation,

i.e. that H is a direct sum of Hilbert spaces coming from irreducible representations (see [Ped79] 3.13.1) \hat{A} of A , one for each unitary equivalence class in \hat{A} . Equivalently, we assume that A is identified with a C*-subalgebra of $\mathcal{B}(H)$ such that every pure state (i.e. extreme point of A_+^{*1} – see [Ped79] 3.10.1) on A is of the form ϕ_v (where $\phi_v(a) = \langle av, v \rangle$, for all $a \in A$) for some unique $v \in H$. This means that open projections distinguish hereditary C*-subalgebras in the sense that $B = A_{p_B}$, for all $B \in \mathcal{H}(A)$. For we certainly have $B \subseteq A_{p_B}$, and if this inclusion were strict then, by [Ped79] Lemma 3.13.5, we would have a pure state ϕ on A with $B \subseteq \phi^\perp \not\subseteq A_{p_B}$, where

$$(2.3) \quad \phi^\perp = \{a \in A : \phi(a^*a) = \phi(aa^*) = 0\}$$

which is determined by some $v \in \mathcal{R}(p_{A_{p_B}} - p_B)$, a contradiction. Thus

$$\mathcal{P}(A'')^\circ \cong \mathcal{H}(A), \quad \text{via } p \mapsto A_p \text{ and } B \mapsto p_B.$$

So any order theoretic question or result about $\mathcal{H}(A)$ has an equivalent formulation in $\mathcal{P}(A'')^\circ$, and an equivalent dual formulation in $\overline{\mathcal{P}(A'')}$, and we will often find it convenient to work with $\mathcal{P}(A'')^\circ$ or $\overline{\mathcal{P}(A'')}$ instead.

3. COCOMPACTNESS AND COMPACTNESS

Definition 1. \mathbb{P} is *compact* if $\forall S \subseteq \mathbb{P}, \bigvee S = 1 \Rightarrow \bigvee F = 1$, for some finite $F \subseteq S$.

Note that X is compact precisely when $\mathcal{P}(X)^\circ$ is compact by the above definition.⁴ And any locally compact X is compact precisely when $C_0(X)$ is unital. For this reason it is often said that unital C*-algebras are non-commutative analogs of compact topological spaces. With the above definition we can make this more precise and identify unitality of A purely from the order structure of $\mathcal{H}(A)$.

Proposition 1. A is unital precisely when $\mathcal{H}(A)$ is compact.

Proof. If A is unital then 1 is q-compact, in the terminology of [Ake71], and hence $\mathcal{H}(A)$ is compact, by (the order theoretic dual of) [Ake71] Theorem II.7.

Now assume A is not unital. If $A = A_a$, for some $a \in A_+$, then 0 is a limit point of $\sigma(a)$, otherwise $a_{\{0\}}^\perp$ would be a unit in A . Thus $A_{a_{(\epsilon_n, \infty)}}$ is a strictly increasing

⁴which is standard in point-free topology – see [PP12] Ch VII. More generally, in lattice theory an element $p \in \mathbb{P}$ is said to be *compact* if, $\forall S \subseteq \mathbb{P}, p \leq \bigvee S \Rightarrow p \leq \bigvee F = 1$, for some finite $F \subseteq S$. So \mathbb{P} is a compact poset precisely when 1 is a compact element. However, this definition of a compact element only identifies the compact open subsets, and when it comes to open set lattices it is rather the cocompact sets we are most interested in. Thus we give a different order theoretic definition of cocompactness in Definition 2, and define compact projections in Definition 3 in the algebraic way more standard in C*-algebra theory.

sequence in $\mathcal{H}(A)$, for some strictly decreasing $\epsilon_n \rightarrow 0$, with $\bigvee A_{a_{(\epsilon_n, \infty)}} = A$, i.e. $\mathcal{H}(A)$ is not compact. On the other hand, if $A \neq A_a$, for any $a \in A_+$, then $(A_a)_{a \in A_+}$ is an upwards directed subset of $\mathcal{H}(A)$ (as $A_{a+b} = A_a \vee A_b$ for all $a, b \in A_+$) with no maximum, even though $A = \bigvee_{a \in A_+} A_a$, i.e. $\mathcal{H}(A)$ is again not compact. \square

Definition 2. We call $p \in \mathbb{P}$ \bigvee -cocompact if $\forall S \subseteq \mathbb{P}, p \vee \bigvee S = 1 \Rightarrow p \vee \bigvee F = 1$, for some finite $F \subseteq S$.

Thus \mathbb{P} is compact precisely when 0 is \bigvee -cocompact by the above definition. Moreover, $O \in \mathcal{P}(X)^\circ$ is \bigvee -cocompact precisely when $X \setminus O$ is a compact subset of X . We also have a more algebraic notion of compactness. Specifically, when X is locally compact and we identify $C_0(X)''$ with $B(X) = (\text{all arbitrary bounded functions from } X \text{ to } \mathbb{C})$, we see that any closed $p \in \mathcal{P}(B(X))$ is the characteristic function of a compact subset of X precisely when $ap = p$, for some $a \in C_0(X)_+^1$. This motivates the following standard definition.

Definition 3. We call $p \in \overline{\mathcal{P}(A'')}$ compact when $ap = p$, for some $a \in A$.

In [Ake71] Theorem II.7 it was shown that p is \bigvee -cocompact in $\mathcal{P}(A'')^\circ$ whenever p^\perp is compact, and [Ake71] Conjecture II.2 predicted that the converse holds (even among arbitrary regular $p \in \mathcal{P}(A'')$). The following example refutes this conjecture.

Example 1. Take $P, Q \in \mathcal{P}(M_2)$ with $0 < \|PQ\| < 1$ and let A be the C^* -subalgebra of $C([0, 1], M_2)$ of functions f with $f(0) \in \mathbb{C}P$. Identify A'' in the usual way with all bounded functions f from $[0, 1]$ to M_2 with $f(0) \in \mathbb{C}P$. Define $q \in \mathcal{P}(A'')^\circ$ by $q(0) = 0$ and $q(x) = Q$ otherwise. As $[0, 1]$ is compact and $P \neq Q$, q is \bigvee -cocompact in $\mathcal{P}(A'')^\circ$. But $q^\perp(x) = Q^\perp \neq P$, for all $x > 0$, so q^\perp is not compact.

On the other hand, in this example r^\perp is compact, where $r(0) = 0$ and $r(x) = P^\perp$ otherwise. Any auto-homeomorphism h of $\mathcal{P}(M_2)$ leaving 0, P and 1 fixed gives rise to an order automorphism θ_h of $\mathcal{P}(A'')^\circ$ defined by $\theta_h(p)(x) = h(p(x))$. If we further require that $h(P^\perp) = Q$ then $\theta_h(r) = q$ and hence θ_h does not preserve compactness (more precisely, it does not preserve the property ' p^\perp is compact'). Thus the compact elements of $\overline{\mathcal{P}(A'')}$ do not even admit any order theoretic characterization.

However, the injection of a little more algebra allows for a partial verification of [Ake71] Conjecture II.2.

Proposition 2. Any $p \in \mathcal{P}(\mathcal{M}(A))$ is \bigvee -cocompact in $\mathcal{P}(A'')^\circ$ iff p^\perp is compact.

Proof. This is essentially the same as the proof of Proposition 1. Specifically, the 'if' part follows from [Ake71] Theorem II.7 (or alternatively one can use the correspondence $q \leftrightarrow \{a \in A_+^1 : aq = q\}$ between non-zero compact projections and proper norm filters, and note that a directed union of norm centred subsets is again norm centred and hence contained in a proper norm filter – see [Bic13b]). While if $p \in \mathcal{P}(\mathcal{M}(A)) = \mathcal{P}(A'')^\circ \cap \overline{\mathcal{P}(A'')}$ and p^\perp is not compact then we obtain (an increasing sequence or) upwards directed $S \subseteq \mathcal{P}(A'')^\circ$ with no maximum and $\bigvee S = p^\perp$. Thus $p \vee \bigvee S = 1$ even though $p \vee \bigvee F \neq 1$ for any finite $F \subseteq S$, i.e. p is not \bigvee -cocompact. \square

4. SEMICOMPLEMENTS AND STRONG ORTHOGONALITY

Definition 4. p is a \wedge -semicomplement of q in \mathbb{P} when $p \wedge q = 0$.

In $\mathcal{P}(X)^\circ$, we see that N and O are \wedge -semicomplements precisely when they are disjoint, which means that $B_O B_N = B_O C^b(X) B_N = \{0\}$. More generally, we define the *orthogonality* \perp and *strong orthogonality* ∇ relations on $\mathcal{H}(A)$ by

$$B \perp C \Leftrightarrow BC = \{0\} \quad \text{and} \quad B \nabla C \Leftrightarrow BAC = \{0\}.$$

For all $B, C \in \mathcal{H}(A)$, we immediately see that

$$B \nabla C \Rightarrow B \perp C \Rightarrow B \cap C = \{0\},$$

and these implications can not be reversed for general non-commutative A . In fact, like with compactness, \perp does not even admit any order theoretic characterization in $\mathcal{H}(A)$. For example, any permutation of $\mathcal{H}(M_2)$ leaving $\{0\}$ and M_2 fixed is an order isomorphism, even though many of these do not preserve the \perp relation. However, we can obtain order theoretic characterizations of ∇ , which is the primary goal of this section.

For this, it turns out to be useful to examine the \vee -semicomplement (the notion dual to a \wedge -semicomplement) structure of $\mathcal{H}(A)$ in more detail.

Definition 5. p is \vee -separated from q in \mathbb{P} if p has a \vee -semicomplement r with $q \leq r < 1$. We call p *subfit* if p is \vee -separated from every $q \not\leq p$. We call \mathbb{P} itself subfit when every $p \in \mathbb{P}$ is subfit.

The term ‘subfit’ comes from [PP12] Ch V §1, at least with reference to entire posets (rather than individual elements), where it is considered as an analog in point-free topology of the T_1 separation axiom. To see why, we introduce atoms.

Definition 6. An *atom* of \mathbb{P} is a minimal element of $\mathbb{P} \setminus \{0\}$. We call $D \subseteq \mathbb{P}$ \vee -dense when $p = \bigvee \{q \in D : q \leq p\}$, for all $p \in \mathbb{P}$. We call \mathbb{P} *atomistic* when the atoms are \vee -dense in \mathbb{P} .

Dually, we define *coatom*, \bigwedge -dense, and *coatomistic*. If X is a T_1 topological space then $\mathcal{P}(X)^\circ$ is coatomistic and hence subfit. Indeed, if \mathbb{P} is coatomistic and $p \not\leq q$ then, as q is the infimum of all coatoms above it, there must be some coatom $r \geq q$ with $r \not\leq p$ and hence $p \vee r = 1$. So $\mathcal{H}(A)$ is coatomistic and hence subfit when A is commutative and, in fact, this easily generalizes to non-commutative A .

Proposition 3. $\mathcal{H}(A)$ is coatomistic.

Proof. If $B, C \in \mathcal{H}(A)$ and $B \not\leq C$ then there is a pure state ϕ on A with $B \not\subseteq \phi^\perp \supseteq C$ (see (2.3)). As ϕ is pure, ϕ^\perp is a coatom in $\mathcal{H}(A)$, by [Ped79] Proposition 3.13.6. Thus any element of $\mathcal{H}(A)$ below all coatoms greater than C is below C , i.e. C is the infimum of all such coatoms. As C was arbitrary, the coatoms are \bigwedge -dense in $\mathcal{H}(A)$. \square

For another topological property related to coatoms, we introduce the following.

Definition 7. We call $p \in \mathbb{P}$ \wedge -irreducible if $p = q \wedge r \Rightarrow p \in \{q, r\}$, for all $q, r \in \mathbb{P}$.

Every coatom in \mathbb{P} is \wedge -irreducible. Also $X \setminus \overline{\{x\}}$ is \wedge -irreducible in $\mathcal{P}(X)^\circ$, for all $x \in X$. We call X *sober* if $\mathcal{P}(X)^\circ$ has no other \wedge -irreducibles apart from X . Among T_0 spaces, T_1 +sobriety is actually a property of the lattice $\mathcal{P}(X)^\circ$,⁵ specifically

X is T_1 and sober $\Leftrightarrow X$ is T_0 and every \wedge -irreducible in $\mathcal{P}(X)^\circ$ is a coatom or X .

⁵even though neither T_1 nor sobriety is, individually, such lattice property (see [PP12] I.3.1)

It would be interesting to know if this also holds in $\mathcal{H}(A)$.

Question 1. *Is every \wedge -irreducible in $\mathcal{H}(A)$ a coatom?*

On the other hand, \wedge -irreducibles in $\mathcal{I}(A)$ are much more well-known. Indeed, they are precisely the *prime* $I \in \mathcal{I}(A)$, i.e. satisfying $aIb \subseteq I \Rightarrow a \in I$ or $b \in I$, for all $a, b \in A$ (see [Ped79] 3.13.7). We also call $I \in \mathcal{I}(A)$ *primitive* if I is the largest element of $\mathcal{I}(A)$ contained in some coatom $B \in \mathcal{H}(A)$, which is equivalent saying I is the kernel of some $\pi \in \hat{A}$ (see [BRVdB89]). Coatoms in $\mathcal{I}(A)$ are usually just called maximal, and we have the following relationships between these concepts in $\mathcal{I}(A)$ (see [Ped79] Proposition 3.13.10)

$$\text{maximal} \Rightarrow \text{primitive} \Rightarrow \text{prime}.$$

Note that none of these implications can be reversed in general, e.g. $\{0\}$ is primitive but not maximal in $\mathcal{B}(H)$ when H is infinite dimensional, while $\{0\}$ is prime but not primitive for the A constructed in [Wea03].

An important property of (arbitrary) $I \in \mathcal{I}(A)$ we will need is the following.

Proposition 4. *For $B, C \in \mathcal{H}(A)$ and $I \in \mathcal{I}(A)$, $I \wedge (B \vee C) = (I \wedge B) \vee (I \wedge C)$.*

Proof. Equivalently, we need to prove that $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ in $\mathcal{P}(A'')^\circ$, whenever $p \in A'$. But \vee agrees in $\mathcal{P}(A'')^\circ$ and $\mathcal{P}(A'')$, as does \wedge for commuting projections, so it suffices to verify the formula in $\mathcal{P}(A'')$. As $q = pq + p^\perp q = pq \vee p^\perp q$ and $r = pr + p^\perp r = pr \vee p^\perp r$, we have

$$p \wedge (q \vee r) = p((pq \vee pr) + (p^\perp q \vee p^\perp r)) = pq \vee pr = (p \wedge q) \vee (p \wedge r).$$

□

To accurately describe some equivalents of the ∇ relation below, we introduce some more notation. Firstly, let \oplus denote the usual (interior) direct sum of vector spaces, so $A = B \oplus C$ means B and C are complementary in the lattice of (arbitrary) subspaces of A . Also define polarities (i.e. order reversing operations) $^\perp$ and $^\nabla$ on $\mathcal{P}(A)$ as in [Bic14a] by

$$B^\perp = \{a \in A : \forall b \in B (ba = 0 = ba^*)\} \quad \text{and} \quad B^\nabla = \{a \in A : \forall b \in B (bAa = \{0\})\}.$$

The elements of $\mathcal{P}(A)^\perp = \{B^\perp : B \in \mathcal{P}(A)\}$ and $\mathcal{P}(A)^\nabla = \{B^\nabla : B \in \mathcal{P}(A)\}$ are called **-annihilators* and *annihilator ideals* respectively.

Theorem 1. *For $B, C \in \mathcal{H}(A)$, the following are equivalent.*

- (1) $B \nabla C$.
- (2) $B^{\nabla \nabla} \cap C^{\nabla \nabla} = \{0\}$.
- (3) $\overline{\text{span}}(ABA) \cap C = \{0\}$.
- (4) $ABA \cap C = \{0\}$.
- (5) $B \vee C = B \oplus C$.
- (6) $D = (B \vee D) \wedge (C \vee D)$, for all $D \in \mathcal{H}(A)$.
- (7) $B \wedge D = B \wedge (C \vee D)$, for all $D \in \mathcal{H}(A)$.
- (8) Every \vee -semicomplement of C in $\mathcal{H}(A)$ contains B .
- (9) Every \wedge -irreducible in $\mathcal{H}(A)$ contains B or C .
- (10) Every coatom in $\mathcal{H}(A)$ contains B or C .
- (11) Every prime $I \in \mathcal{I}(A)$ contains B or C .
- (12) Every primitive $I \in \mathcal{I}(A)$ contains B or C .
- (13) $aa^* \in B$ and $a^*a \in C \Rightarrow a = 0$, for all $a \in A$.

Proof. We immediately see that (2) \Rightarrow (3) \Rightarrow (4), (6) \Rightarrow (7) \Rightarrow (8) \Rightarrow (10), (6) \Rightarrow (9) \Rightarrow (10) and (11) \Rightarrow (12). The rest of the equivalences are proved as follows.

- (1) \Rightarrow (2) As $B \nabla C$, we have $B \subseteq C^\nabla$ and hence $C^{\nabla\nabla} \subseteq B^\nabla$ which, as $B^\nabla \cap B^{\nabla\nabla} = \{0\}$, means $B^{\nabla\nabla} \cap C^{\nabla\nabla} = \{0\}$.
- (4) \Rightarrow (1) Take $a \in BAC$. Then $a^*a \in ABA \cap C = \{0\}$ so $a = 0$.
- (3) \Rightarrow (8) Let $I = \overline{\text{span}}(ABA)$. If D is a \vee -semicomplement of C in $\mathcal{H}(A)$ then, by Proposition 4, $I = I \wedge A = I \wedge (C \vee D) = (I \wedge C) \vee (I \wedge D) = I \wedge D$, so $B \subseteq I \subseteq D$.
- (8) \Rightarrow (6) As $\mathcal{H}(A)$ is subfit, by Proposition 3, $\mathcal{H}(A)$ is SSC*, in the terminology of [MM70], where this implication appears as Theorem (4.18) $(\beta) \Rightarrow (\gamma)$.
- (10) \Rightarrow (12) Say we had some primitive ideal which contained neither B nor C , i.e. we have $\pi \in \hat{A}$ with $\pi[B] \neq \{0\} \neq \pi[C]$. Thus we have some $b \in B$, $c \in C$ and $v \in H_\pi$ with $\pi(b)v \neq 0 \neq \pi(c)v$. Indeed, if $\pi(p_B)\pi(p_C) \neq 0$ then we can pick $v \in \mathcal{R}(\pi(p_B))$ (or $v \in \mathcal{R}(\pi(p_C))$), while if $\pi(p_B)\pi(p_C) = 0$ then we can set $v = x + y$ for any $x \in \mathcal{R}(\pi(p_B)) \setminus \{0\}$ and $y \in \mathcal{R}(\pi(p_C)) \setminus \{0\}$. As π is irreducible, $\phi_v = \langle \pi(\cdot)v, v \rangle$ is a pure state, and hence ϕ_v^\perp is a coatom containing neither B nor C .
- (12) \Rightarrow (13) By (12), $\pi(aa^*) = 0$ or $\pi(a^*a) = 0$, and hence $\pi(a) = 0$, for every $\pi \in \hat{A}$. As π was arbitrary, $a = 0$.
- (13) \Rightarrow (1) If $a \in BAC$ then $aa^* \in BAB \subseteq B$ and $a^*a \in CAC \subseteq C$ so $a = 0$.
- (1) \Rightarrow (5) As B and C are C*-subalgebras of A with $B \perp C$, $B \oplus C$ is also a C*-subalgebra of A . As B and C are hereditary and $BAC = \{0\}$, we also have $(B+C)A(B+C) \subseteq BAB + CAC \subseteq B+C$. Thus $B \oplus C$ is also hereditary, by [Bla13] Corollary II.5.3.9.
- (5) \Rightarrow (13) Take $a \in A$ with $aa^* \in B$ and $a^*a \in C$, so $a \in B \vee C = B \oplus C$, i.e. $a = b + c$, for some $b \in B$ and $c \in C$. Hence

$$aa^* - ab^* - ba^* + bb^* = (a - b)(a^* - b^*) = cc^* \in B \cap C = \{0\}.$$

Thus $c = 0$ and, likewise, $b = 0$ and hence $a = b + c = 0$.

- (1) \Rightarrow (11) Assume $B \nabla C$, which we already know is equivalent to $B^{\nabla\nabla} \nabla C^{\nabla\nabla}$. Say $I \in \mathcal{I}(A)$ contains neither B nor C , so $I \subsetneq I \vee B^{\nabla\nabla}, I \vee C^{\nabla\nabla}$, even though $I = (I \vee B^{\nabla\nabla}) \wedge (I \vee C^{\nabla\nabla})$, by (6), i.e. I is not \wedge -irreducible in $\mathcal{I}(A)$.

□

Thus (6), (7), (8), (9) and (10) give us purely order theoretic characterizations of ∇ (and in lattice theory (7) is often also denoted by ∇ and called the ‘del’ relation). The same could be said of (2), (3), (11) and (12) once it is known that ideals and annihilator ideals have order theoretic characterizations, as shown in §§6 and 7. Alternatively, note that B^∇ is the maximum C in $\mathcal{H}(A)$ or $\mathcal{I}(A)$ with $B \nabla C$, so the order-theoretic characterizations of ∇ here yield order theoretic characterizations of annihilator ideals. Also, it would be interesting to know if $B \nabla C$ even when B and C are complementary in the lattice of *closed* subspaces of $B \vee C$, i.e. whether (5) can be weakened to $B \vee C = \overline{B + C}$ and $B \cap C = \{0\}$.

Incidentally, one might also consider the algebraic relation $A = B + C$ to be something of a dual to ∇ . Indeed, it agrees with the \vee -semicomplement relation $A = B \vee C$ when A is commutative (as then B and C are ideals so $B \vee C = B + C$ – see [Ped79] Corollary 1.5.8), but is significantly stronger for non-commutative A .

Question 2. *Can some dual to Theorem 1 be proved for the relation $A = B + C$?*

5. COMPLEMENTS AND PROJECTIONS

Definition 8. We call $p, q \in \mathbb{P}$ *complementary* when

$$p \wedge q = 0 \quad \text{and} \quad p \vee q = 1.$$

The complements in $\mathcal{P}(X)^\circ$ are precisely the clopen (i.e. closed and open) subsets. Also, the projections in $C^b(X)(\cong \mathcal{M}(C_0(X)))$ when X is locally compact are precisely the characteristic functions of clopen subsets of X . Thus complements in $\mathcal{H}(A)$ correspond to projections in $\mathcal{M}(A)$ whenever A is commutative. The motivating question for this section is whether this extends to non-commutative A . Phrased in terms of $\mathcal{P}(A'')^\circ \cong \mathcal{H}(A)$, the following result immediately provides a partial answer.

Proposition 5. *If $p \in \mathcal{P}(\mathcal{M}(A))$ then p and p^\perp are complementary in $\mathcal{P}(A'')^\circ$.*

Proof. As $\mathcal{P}(\mathcal{M}(A)) = \{p \in \mathcal{P}(A'') : p^\circ = \bar{p}\}$, by [Ped79] Theorem 3.12.9, $p^\perp \in \mathcal{P}(A'')^\circ$. But p and p^\perp are complementary in $\mathcal{P}(A'')$ and so certainly in $\mathcal{P}(A'')^\circ$. \square

We can also prove the converse, but only when one of the projections is compact. The proof also requires the following elementary results.

Denote the spectral projection in A'' of $a \in A_+$ corresponding to $S \subseteq \mathbb{R}_+$ by a_S . For any $a, b \in A_+$, $\epsilon > 0$, and $v \in \mathcal{R}((a+b)_{[0, \epsilon^3]})$,

$$\begin{aligned} \epsilon \|a_{(\epsilon, \infty)} v\|^2 &= \epsilon \langle a_{(\epsilon, \infty)} v, v \rangle \leq \langle av, v \rangle \leq \langle (a+b)v, v \rangle \leq \epsilon^3 \langle v, v \rangle = \epsilon^3 \|v\|^2, \text{ so} \\ (5.1) \quad \|(a+b)_{[0, \epsilon^3]} a_{(\epsilon, \infty)}\| &\leq \epsilon. \end{aligned}$$

Also, for any $p_1, \dots, p_n \in \mathcal{P}(A)$, $\epsilon > 0$ and unit $v \in H$ such that $\|p_k^\perp v\| \leq \epsilon$, for all $k \leq n$, we have,

$$(5.2) \quad \|p_1 \dots p_n\| \geq \|p_1 \dots p_n v\| \geq \|p_1 \dots p_{n-1} v\| - \epsilon \geq \dots \geq 1 - n\epsilon.$$

Theorem 2. *If $p, q \in \mathcal{P}(A'')^\circ$ are complementary and q^\perp is compact then $p \in A$ and $\|pq\| = \|p^\perp q^\perp\| < 1$.*

Proof. Assume, to the contrary, that $\|pq\| = 1$. As $\sigma(pq) = \sigma(pqp)$, by [HO88], if $1 > \sup(\sigma(pq) \setminus \{1\})$ then $0 \neq p \wedge q \in \mathcal{P}(A'')^\circ$, by [Ake69] Theorem II.7⁶, contradicting the assumption that $(p \wedge q)^\circ = 0$. But $\sigma(pq) \setminus \{0, 1\} = \sigma(p^\perp q^\perp) \setminus \{0, 1\}$, by [Bic13c] §2.2, so if $1 = \sup(\sigma(pq) \setminus \{1\}) = \sup(\sigma(p^\perp q^\perp) \setminus \{1\})$ then $\|p^\perp q^\perp\| = 1$. And if $\|p^\perp q^\perp\| = 1$ then we have a state ϕ on A'' with $\phi(p^\perp) = 1 = \phi(q^\perp)$, by [Bic13b] Theorem 2.2. As q^\perp is compact, $q^\perp \leq a$ and hence $\phi(a) = 1$, for some $a \in A_+^1$. Thus ϕ restricts to a state on A , so defining ϕ^\perp as in (2.3) yields $p, q \leq p_{\phi^\perp} < 1$ (note p is open so $0 < p = \bigvee A_{p+}^1$ and hence $\phi(a) = 0$ for some $a \in A_+ \setminus \{0\}$, i.e. ϕ is not faithful on A), contradicting the assumption that $p \vee q = 1$. Thus $\|pq\| = \sqrt{\sup(\sigma(pq))} = \sqrt{\sup(\sigma(p^\perp q^\perp))} = \|p^\perp q^\perp\| < 1$.

Now suppose that $p \notin A$, so $b_{(\epsilon, \infty)} < p$, for all $b \in A_{p+}$ and $\epsilon > 0$. We claim that, moreover, $((q \vee b_{(\epsilon, \infty)})^\perp)_{b \in A_{p+}, \epsilon > 0}$ is norm centred (see [Bic13b] Definition 2.1). To see this, take $b_1, \dots, b_n \in A_{p+}$ and $\epsilon_1, \dots, \epsilon_n > 0$. Let $b = b_1 + \dots + b_n \in A_{p+}$ and, for any $\epsilon > 0$, let $\delta = \min(\epsilon^3, \epsilon_1^3, \dots, \epsilon_n^3)$. By (5.1),

$$\|b_{[0, \delta]}(b_k)_{(\epsilon_k, \infty)}\| \leq \epsilon,$$

⁶Actually, there is slight oversight in the proof of [Ake69] Theorem II.7. Specifically, in paragraph 2 line 2, q is replaced with $q_0 = q - p \wedge q$, which is fine until line 4 from the bottom, where we must revert back to the original q .

for all $k \leq n$. Thus, by the inequality in line 4 of the proof [Bic13c] Lemma 2.7 (where $P = q$, $Q = (b_k)_{(\epsilon_k, \infty)}$ and $R = b_{(\delta, \infty)}$),

$$\|(q \vee b_{(\delta, \infty)})^\perp (q \vee (b_k)_{(\epsilon_k, \infty)})\| \leq \epsilon / \sqrt{1 - \|pq\|^2}.$$

As $\|pq\| < 1$, we have $\mathcal{R}(r \vee q) = \mathcal{R}(r) + \mathcal{R}(q)$, for any $r \leq p$, i.e. the supremum of these projections is just the supremum of the corresponding subspaces. As we also know that subspaces of a vector space are modular,

$$(b_{(\delta, \infty)} \vee q) \wedge p = b_{(\delta, \infty)} \vee (q \wedge p) = b_{(\delta, \infty)} < p$$

so $b_{(\delta, \infty)} \vee q \neq 1$. Thus, we can take unit $v \in \mathcal{R}(q \vee b_{(\delta, \infty)})^\perp$ and (5.2) yields

$$\|(q \vee (b_1)_{(\epsilon_1, \infty)})^\perp \dots (q \vee (b_n)_{(\epsilon_n, \infty)})^\perp\| \geq 1 - n\epsilon / \sqrt{1 - \|pq\|^2}.$$

As ϵ was arbitrary, we in fact have $\|(q \vee (b_1)_{(\epsilon_1, \infty)})^\perp \dots (q \vee (b_n)_{(\epsilon_n, \infty)})^\perp\| = 1$ which, as b_1, \dots, b_n and $\epsilon_1, \dots, \epsilon_n$ were arbitrary, shows that $((q \vee b_{(\epsilon, \infty)})^\perp)_{b \in A_{p^+}, \epsilon > 0}$ is indeed norm centred. Thus we have a state ϕ on A'' with $\phi((q \vee b_{(\epsilon, \infty)})^\perp) = 1$, for all $b \in A_{p^+}$ and $\epsilon > 0$, which means that $\phi[A_p] = \{0\} = \phi[A_q]$. Again, as q^\perp is compact, ϕ restricts to a state on A so $p, q \leq p_{\phi^\perp} < 1$, contradicting $p \vee q = 1$. \square

In particular, if A is unital then any complementary $p, q \in \mathcal{P}(A'')^\circ$ must lie in A and $\mathcal{P}(A)$ is a (first order) definable subset of the lattice $\mathcal{P}(A'')^\circ$ (the weaker statement that $\mathcal{P}(A)$ can be determined from the (non-first order) lattice structure of $\mathcal{P}(A'')^\circ$ follows already from Proposition 1). One might conjecture that even when A is non-unital, complementary $p, q \in \mathcal{P}(A'')^\circ$ must lie in $\mathcal{M}(A)$. The following example shows this to be false.

Example 2. Take $P, Q \in \mathcal{P}(M_2)$ with $0 < \|PQ\| < 1$ and let A be the C*-subalgebra of $C([0, 1], M_2)$ of functions f with $f(0) \in \mathbb{C}P$ and $f(1) \in \mathbb{C}Q$. Identify A'' in the usual way with all bounded functions f from $[0, 1]$ to M_2 with $f(0) \in \mathbb{C}P$ and $f(1) \in \mathbb{C}Q$. Define $p, q \in \mathcal{P}(A'')^\circ$ by $p(1) = 0$, $p(x) = P$ otherwise, $q(0) = 0$ and $q(x) = Q$ otherwise. Then p and q are complementary (not just in $\mathcal{P}(A'')^\circ$ but even in $\mathcal{P}(A'')$) even though neither p nor q is in $\mathcal{M}(A)$.

Question 3. *Is there any algebraic characterization (or anything non-trivial that can be said) of complements in $\mathcal{H}(A)$ when A is not unital?*

Strengthening the hypothesis in Proposition 5 also yields uniqueness.

Proposition 6. *If $p \in \mathcal{P}(\mathcal{M}(A) \cap A')$, p^\perp is the unique complement of p in $\mathcal{P}(A'')^\circ$.*

Proof. In $\mathcal{P}(A'')^\circ$, we have $q \wedge r = qr$, whenever $qr = rq$, and $q \vee r = q + r$, whenever $qr = 0$. Thus $p \wedge p^\perp = pp^\perp = 0$ and $p \vee p^\perp = p + p^\perp = 1$, i.e. p^\perp is a complement of p . Moreover, if q is another complement of p then $pq = p \wedge q = 0$ so $p + q = p \vee q = 1$, i.e. $q = p^\perp$. \square

Again, we can prove the converse for compact projections.

Corollary 1. *If compact p is a unique complement of $q \in \overline{\mathcal{P}(A'')}$ then $p \in A \cap A'$.*

Proof. By Theorem 2, $q^\perp \in A$ so $q \in \mathcal{M}(A)$. By Proposition 5, q^\perp is a complement of q which, if p is the unique complement of q , gives $p = q^\perp \in A$. Assuming $p \notin A'$, we would have $a \in A_+^1$ with $ap \neq pa$. As $\|a\| \leq 1 < \pi$, $a = f(e^{ia})$, where f is a continuous function on $\mathbb{T} = \mathbb{C}^1 \setminus \mathbb{C}^{1^\circ}$, so p does not commute with e^{ia} either, i.e. $e^{ia}pe^{-ia} \neq p$. Multiplying a by some $\epsilon > 0$ if necessary, we can also make $\|1 - e^{ia}\| <$

$\frac{1}{2}$ so $\|p - e^{ia}pe^{-ia}\| < 1$, which means $\|qe^{ia}pe^{-ia}\| = \|q^\perp(e^{ia}pe^{-ia})^\perp\| < 1$. Thus $e^{ia}pe^{-ia}$ is a complement of q in $\mathcal{P}(A'')$ and so certainly in $\overline{\mathcal{P}(A'')}$, i.e. q has more than one complement, a contradiction (see also [Oza14]). \square

So if A is unital and p is or has a unique complement in $\mathcal{P}(A'')^\circ$ then $p \in A$. Unlike Theorem 2, this might lead to a non-compact version of Corollary 1.

Question 4. *Is $p \in \mathcal{M}(A) \cap A'$ whenever p is/has a unique complement in $\mathcal{P}(A'')^\circ$?*

6. PSEUDOCOMPLEMENTS AND ANNIHILATOR IDEALS

Definition 9 ([Bir67] Ch V §8). p is a \wedge -pseudocomplement of q in \mathbb{P} when

$$p = \bigvee \{r : q \wedge r = 0\} \quad \text{and} \quad p \wedge q = 0.$$

In other words, a \wedge -pseudocomplement is a maximum \wedge -semicomplement. Every $O \in \mathcal{P}(X)^\circ$ has a \wedge -pseudocomplement given by $(X \setminus O)^\circ$. So \wedge -pseudocomplements in $\mathcal{P}(X)^\circ$ are precisely the interiors of closed sets, i.e. the *regular* open sets. Also,

$$\mathcal{P}(C_0(X))^\perp = \mathcal{P}(C_0(X))^\nabla = \{B_O : O \text{ is regular open}\},$$

for any locally compact X . Thus, when A is commutative, we have natural bijections between $*$ -annihilators/annihilator ideals and \wedge -pseudocomplements in $\mathcal{H}(A)$. A quick check of elementary non-commutative A yields $*$ -annihilators that are not \wedge -pseudocomplements, which makes it only reasonable to conjecture that \wedge -pseudocomplements in $\mathcal{H}(A)$ are precisely the annihilator ideals. This is the central result we prove in this section.

Proposition 7. *If $I \in \mathcal{H}(A)$ is an ideal, I^\perp is a \wedge -pseudocomplement of I and*

$$I \text{ is a } \wedge\text{-pseudocomplement} \quad \Leftrightarrow \quad I = I^{\perp\perp}.$$

Proof. Say $B \in \mathcal{H}(A)$ and $B \cap I = \{0\}$. If $B \not\subseteq I^\perp$, then there exists $b \in B$ and $a \in I$ such that $ab \neq 0$ and hence $0 \neq b^*a^*ab \in B \cap I$, as B is hereditary and I is an ideal, a contradiction. As B was arbitrary, I^\perp is a \wedge -pseudocomplement of I . In particular, as I^\perp is also an ideal, $I^{\perp\perp}$ is a \wedge -pseudocomplement of I^\perp . And if I is a \wedge -pseudocomplement of some $C \in \mathcal{H}(A)$ then $C \subseteq I^\perp$. For, if not, we could find $a \in I$ and $c \in C$ such that $ac \neq 0$ and hence $0 \neq c^*a^*ac \in I \cap C$, as I is an ideal and C is hereditary, contradicting $I \cap C = \{0\}$. Thus we have $I \subseteq I^{\perp\perp} \subseteq C^\perp \subseteq I$, the last inclusion coming from the defining property of a \wedge -pseudocomplement (and the fact $C \cap C^\perp = \{0\}$), i.e. $I = I^{\perp\perp}$. \square

Note that $B^\perp = B^\nabla$ whenever B is a (right) ideal, so the above result implies that every annihilator ideal is a \wedge -pseudocomplement, and we now set about proving the converse. This requires the following lemma, which will come in handy again later on. It gives a simple algebraic description of the restriction of a certain Sasaki projection (see [Kal83] §5 p99) on $\mathcal{H}(A_a)$. Firstly, let $u = u_a \in A'$ denote the partial isometry coming from the polar decomposition of $a \in A$ (see [Ped79] Proposition 2.2.9), so $u^*u = (a^*a)_{\{0\}}^\perp$, $uu^* = (aa^*)_{\{0\}}^\perp$ and $a = u|a| = |a^*|u$.

Lemma 1. *If $a \in A$ and $a^2 = 0$ then $u = u_a \in \mathcal{M}(A_a)$ and, for any $B \in \mathcal{H}(vA_a v^*)$, where $v = v_a = \frac{1}{\sqrt{2}}(u + u^*u)$, we have*

$$(6.1) \quad \overline{a^*Aa} \cap (B \vee \overline{aAa^*}) = v^*Bv.$$

Proof. For continuous f on \mathbb{R}_+ with $f(0) = 0$, we have $uf(a^*a), f(aa^*)u \in A_a$ (see [Cum77] Proposition 1.3). For such $(f_n) \uparrow \chi_{(0,\infty)}$ (pointwise) and for any $d \in A_a$, we have $(f_n(a^*a) + f_n(aa^*))d \rightarrow d$ and $d(f_n(a^*a) + f_n(aa^*)) \rightarrow d$ so

$$ud = \lim uf_n(a^*a)d \in A_a \quad \text{and} \quad du = \lim df_n(a^*a)u \in A_a.$$

As $d \in A_a$ was arbitrary, $u \in \mathcal{M}(A_a)$.

Take $p \leq vv^*$. As $\sqrt{2}(1-uu^*)vv^* = \frac{1}{\sqrt{2}}(u^* + u^*u) = v^*$, we have $\sqrt{2}(1-uu^*)p = v^*p$ and hence $v^*pv = 2(1-uu^*)p(1-uu^*) \leq p \vee uu^*$ so $uu^* + v^*pv \leq p \vee uu^*$. Also $v^*uu^*v = \frac{1}{2}u^*u = \frac{1}{2}v^*v$ and hence $vv^*uu^*vv^* = \frac{1}{2}vv^*$ so $puu^*p = \frac{1}{2}p$. But also $v^2 = \frac{1}{2}(u + u^*u) = \frac{1}{\sqrt{2}}v$ and hence $vv^*vvv^* = \frac{1}{\sqrt{2}}vv^*$ so $pvp = \frac{1}{\sqrt{2}}p$ and therefore $pvpvp = \frac{1}{2}p$. Thus $p(uu^* + v^*pv)p = p$, i.e. $p \leq uu^* + v^*pv$. We certainly also have $uu^* \leq uu^* + v^*pv$ and thus $p \vee uu^* = uu^* + v^*pv$. Hence

$$u^*u \wedge (p \vee uu^*) = v^*pv.$$

If p is also open then so is $v^*pv = \bigvee v^*A_{p+}^1v$, thus verifying (6.1). \square

Incidentally, it would be interesting to know if $u_a \in \mathcal{M}(A_a)$ even when $a^2 \neq 0$. Also, an important special case of Lemma 1 occurs when $B = vA_av^*$, in which case (6.1) becomes $\overline{a^*Aa} \cap (vA_av^* \vee \overline{aAa^*}) = \overline{a^*Aa}$, i.e.

$$(6.2) \quad \overline{a^*Aa} \subseteq vA_av^* \vee \overline{aAa^*}$$

Note too that $vv^* = \frac{1}{2}(uu^* + u + u^* + u^*u)$ so, as $u_a^* = u_a^*$, we have $v_av_a^* = v_a^*v_a^*$. Thus, replacing a with a^* in (6.1) we get $\overline{a^*Aa} \cap (B \vee \overline{aAa^*}) = v_a^*Bv_a^*$ and (6.2) becomes

$$(6.3) \quad \overline{aAa^*} \subseteq vA_av^* \vee \overline{a^*Aa}$$

Theorem 3. *If B is a \wedge -pseudocomplement of C in $\mathcal{H}(A)$ then $B = C^\perp = C^\nabla$.*

Proof. As $C \cap C^\perp = \{0\}$, we have $C^\nabla \subseteq C^\perp \subseteq B$, by the definition of \wedge -pseudocomplement. Thus it suffices to prove $B \subseteq C^\nabla$ or, equivalently, $B \nabla C$.

Assume to the contrary that there is some $a \in A \setminus \{0\}$ with $a^*a \in B$ and $aa^* \in C$. We first claim that, by finding a suitable replacement if necessary, we can further assume that $a^2 = 0$. To see this, take $\delta \in (0, \|a\|)$ and let f and g be continuous functions on \mathbb{R}_+ with $f(0) = 0$, $f(r) = 1$, for all $r \geq \delta$, $g(r) = 0$, for all $r \leq \delta$, and $g(r) \in (0, 1)$, for all $r > \delta$ (so $fg = g$). For $u = u_a$, again by [Cum77] Proposition 1.3 we have $d = ug(|a|) = g(|a^*|)u \in A$. Let $e = d - f(|a|)d$ and note that, as $g(|a|)f(|a|) = g(|a|)$, we have $df(|a|) = d$ so $de = 0$ and hence $e^2 = 0$. Thus if there exists $c \in \overline{eAe^*} \cap C \setminus \{0\}$ then we may replace a with ce and we are done. Otherwise, $\overline{eAe^*} \cap C = \{0\}$ and hence $\overline{eAe^*} \subseteq B$, as B is a \wedge -pseudocomplement of C . Note that

$$\begin{aligned} ee^* &= (1 - f(|a|))dd^*(1 - f(|a|)) \\ &= dd^* - f(|a|)dd^* - dd^*f(|a|) + f(|a|)dd^*f(|a|) \\ &\geq dd^* - (dd^*)^2 - f(|a|)^2 + f(|a|)dd^*f(|a|), \end{aligned}$$

(we are using the fact that $x^2 - xy - yx + y^2 = (x - y)^2 \geq 0$, for any $x, y \in A_{sa}$, where here $x = dd^*$ and $y = f(|a|)$) so, as $ee^*, a^*a \in B$,

$$(6.4) \quad dd^* - (dd^*)^2 \leq ee^* + f(|a|)^2 - f(|a|)dd^*f(|a|) \in B.$$

But $dd^* = g(|a^*|)^2$ so $\|d\|^2 < 1$ and hence $0 < dd^* - (dd^*)^2 \in C$, as $aa^* \in C$, contradicting $B \cap C = \{0\}$. Thus the claim is proved.

So, further assuming that $a^2 = 0$, we may consider $D = vA_a v^* \in \mathcal{H}(A)$ as in [Lemma 1](#). If $C \cap D = \{0\}$ then, as B is a \wedge -pseudocomplement of C , we have $D \subseteq B$. But then, by [\(6.3\)](#),

$$\overline{aAa^*} \subseteq D \vee \overline{a^*Aa} \subseteq B,$$

even though $aa^* \in C$, contradicting $B \cap C = \{0\}$. But if $E = C \cap D \neq \{0\}$ then $E \vee \overline{aAa^*} \subseteq C$ even though

$$\{0\} \neq v^*Ev = \overline{a^*Aa} \cap (E \vee \overline{aAa^*}) \subseteq B,$$

again contradicting $B \cap C = \{0\}$. \square

Even when B has no \wedge -pseudocomplement $\mathcal{H}(A)$, we can still prove something about the supremum of all \wedge -semicomplements of B (see [Proposition 8](#)) using the following result, which strengthens [\[Ped79\] Lemma 2.6.3](#).

Let $\mathcal{U}(S)$ denote the unitaries in $S \subseteq \mathcal{M}(A)$, and let $c(a)$ denote the central cover of a in A''_{sa} , as in [\[Ped79\] 2.6.2](#). So if $p \in \mathcal{P}(A'')$ then $c(p)$ is the smallest element of $\mathcal{P}(A'' \cap A')$ with $p \leq c(p)$ and, moreover, $c(p)$ is open whenever p is, in which case $c(p) = \bigvee (\overline{\text{span}}(AA_p A))_+^1$.

Lemma 2. *For every $\epsilon > 0$ and $p \in \mathcal{P}(A'')$, we have $c(p) = \bigvee_{u \in \mathcal{U}(1+A^\epsilon)} upu^*$.*

Proof. If $p_U = \bigvee_{u \in \mathcal{U}(1+A^\epsilon)} upu^* < c(p)$ then we have an irreducible representation π of A such that $\pi(p) \neq 0$ and $\pi(p_U) \neq 1$. Therefore there exist unit vectors $v \in \mathcal{R}(\pi(p)) \subseteq \mathcal{R}(\pi(p_U))$ and $w \in \mathcal{R}(\pi(p_U))^\perp$. Let b be a self-adjoint operator of norm $\pi/2$ on $\text{span}(v, w)$ such that $e^{ib}(v) = w$ and $e^{ib}(w) = v$. By Kadison's transitivity theorem we have $c \in A_{\text{sa}}$ such that $\pi(c)$ agrees with b on $\text{span}(v, w)$ and hence, for all $t \in (0, 1)$, $e^{itc}pe^{-itc} \not\leq p_U$ (because $w \in \mathcal{R}(p \vee e^{itc}pe^{-itc})$ and $p \leq p_U$), contradicting the definition of p_U . \square

Proposition 8. *For any $p \in \mathcal{P}(A'')^\circ$, we have $c(p^{\perp\circ}) \leq \bigvee \{q \in \mathcal{P}(A'')^\circ : p \wedge q = 0\}$.*

Proof. Note that $pp^{\perp\circ} = 0$ so, for all $u \in \mathcal{U}(1+A^{1/3})$, $\|pup^{\perp\circ}u^*\| < 1$, and hence $p \wedge up^{\perp\circ}u^* = 0$. Thus $c(p^{\perp\circ}) \leq r = \bigvee \{q \in \mathcal{P}(A'')^\circ : p \wedge q = 0\}$, by [Lemma 2](#). \square

To see that the inequality here can be strict, see [Example 4](#) below.

This also gives us a simple way of proving a weakened form of [Theorem 3](#).

Corollary 2. *If p has a \wedge -pseudocomplement in $\mathcal{P}(A'')^\circ$ then $p^{\perp\circ} \in A'$.*

Proof. By [Proposition 8](#), $c(p^{\perp\circ}) \leq q$, where q is the \wedge -pseudocomplement of p , so $pc(p^{\perp\circ}) = (p \wedge c(p^{\perp\circ}))^\circ = 0$ and hence $c(p^{\perp\circ}) = p^{\perp\circ}$, i.e. $p^{\perp\circ} \in A'$. \square

7. DISTRIBUTIVITY AND IDEALS

Definition 10. In a lattice \mathbb{P} we call $p \in \mathbb{P}$ \vee -distributive if, for all $q, r \in \mathbb{P}$,

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r).$$

Likewise, we call $p \in \mathbb{P}$ \bigvee -distributive if, for all $S \subseteq \mathbb{P}$,

$$p \wedge \left(\bigvee_{s \in S} s \right) = \bigvee_{s \in S} (p \wedge s).$$

Every $O \in \mathcal{P}(X)^\circ$ is \vee -distributive⁷ and, more generally, it is well-known in quantale theory (see [BVdB86]) that ideals in $\mathcal{H}(A)$ are \vee -distributive (alternatively this can be proved using open projections, as in Proposition 4). Conversely it was shown in [BRVdB89] Proposition 5 that, for liminal A , certain \vee -distributive elements are ideals, where it was also asked what the general \vee -distributive elements in (even non-liminary) A look like. It turns out that the obvious candidate works.

Theorem 4. *If B is \vee -distributive in $\mathcal{H}(A)$ then B is an ideal.*

Proof. Take $B \in \mathcal{H}(A)$ that is not an ideal, so we have $x \in A$ and $y \in B$ with $xy \notin B$. As $y^*x^*xy \in B$, we must have $xyy^*x \notin B$ so, setting $a = xy$, we have $|a|^2 = a^*a \in B$ but $|a^*|^2 = aa^* \notin B$. Define continuous functions (g_n) on \mathbb{R}_+ which uniformly approach the identity and satisfy $g_n^{-1}\{0\} \supseteq [0, \delta_n) \neq \emptyset$, for all $n \in \mathbb{N}$. Then $g_n(|a^*|) \rightarrow |a^*| \notin B$ which, as B is closed, means $g_n(|a^*|) \notin B$, for some n . Set $g = \lambda g_n$ for this n , where $\lambda \in (0, 1/||a||)$, and define f , d and e as in the first paragraph of the proof of Theorem 3. If we had $ee^* \in B$ then $g(|a^*|)^2 = dd^* \in B$, by (6.4), a contradiction. Thus, by replacing a with e if necessary, we may further assume that $a^2 = 0$.

Now set $u = u_a$, $v = v_a$, $C = \overline{aAa^*}$ and $D = vA_av^*$. As $aa^* \notin B$, we have $\phi \in A_+^* \setminus \{0\}$ with $\phi(aa^*) > 0$ and $\phi[B] = \{0\}$. Define $\psi \in A_+^* \setminus \{0\}$ by $\psi(z) = \phi(uzu^*)$. Note ϕ extends uniquely to a normal state on A'' so ψ is well defined (in fact, as A is given the reduced atomic representation, there must be some $h \in H$ with $\phi = \phi_h$ and then we may set $\psi = \phi_{u^*h}$). Note that $(1 - u^*u)v = \frac{1}{\sqrt{2}}u = uv$ so $(1 - u^*u)vzv(1 - u^*u) = uvzv^*u^*$, for any $z \in A$, so $(1 - u^*u)d(1 - u^*u) = udu^*$, for any $d \in D$. Thus if $d \in B \cap D$ then $udu^* \in B$ and hence $\psi(d) = \phi(udu^*) = 0$, i.e. $\psi[B \cap D] = \{0\}$. As $a^2 = 0$, $\psi[C] = \{0\}$ too and hence $\psi[(B \cap C) \vee (B \cap D)] = \{0\}$. But $a^*a \in C \vee D$, by (6.2), and $a^*a \in B$ too even though $\psi(a^*a) = \phi(ua^*au^*) = \phi(u|a||a|u^*) = \phi(aa^*) > 0$ so

$$(B \cap C) \vee (B \cap D) \neq B \cap (C \vee D).$$

□

We can make $\mathcal{H}(A)$ a quantale (see [Mul86]) by defining $B \& C = B \cap \overline{\text{span}}(ABA)$. Moreover, this agrees with the usual quantale structure on the lattice $\mathcal{I}_R(A)$ of closed right ideals given by $I \& J = \overline{\text{span}}(IJ)$, when we identify every $I \in \mathcal{I}_R(A)$ with $I \cap I^* \in \mathcal{H}(A)$ (see [BVdB86] Proposition 2). By Theorem 4, the ideals in $\mathcal{H}(A)$ can be identified purely from the order structure of $\mathcal{H}(A)$, so the lattice $\mathcal{H}(A)$ completely determines the quantale $\mathcal{H}(A)$. As any postliminary A is completely determined by the quantale $\mathcal{H}(A)$, by the theorem at the end of [BRVdB89],⁸ we also get the following strengthening of [BRVdB89] Proposition 5.

Corollary 3. *Any postliminary A is completely determined by the lattice $\mathcal{H}(A)$.*

⁷Indeed, most classical point-set topology can be done, albeit with various subtle differences, in the context of *frames/locales* which are, by definition, just complete (bounded) lattices in which every element is \vee -distributive (see [PP12]).

⁸There seem to be some details missing in the proof of this theorem. Specifically, we do not see how to get an isomorphism of q-spaces, i.e. an algebraic isomorphism of $\mathcal{B}(H)$ and $\mathcal{B}(H')$ identifying open projections, purely from a quantale isomorphism. Hopefully this is just a lack of understanding on our part for, if not, it would put our stronger result Corollary 3 in some doubt.

It would be interesting to know if this can be extended to some broader class of C^* -algebras. Another natural structure to consider on $\mathcal{H}(A)$ would be the Peligrad-Zsidó equivalence relation \sim_{PZ} given in [PZ00], and this might help in distinguishing more C^* -algebras. However, this would still not distinguish between A and A^{op} , even though these can be non-isomorphic C^* -algebras (see [Phi04]). But perhaps $(\mathcal{H}(A), \sim_{\text{PZ}})$ could still be a complete isomorphism invariant within a large class of (Elliot invariant) classifiable C^* -algebras?

Dual to Definition 10, we also have \wedge -distributivity and \bigwedge -distributivity. Moreover, Theorem 4 can also be proved with \wedge -distributivity in place of \vee -distributivity.

Theorem 5. *If B is \wedge -distributive in $\mathcal{H}(A)$ then B is an ideal.*

Proof. If $B \in \mathcal{H}(A)$ is not an ideal then, as in the proof of Theorem 4, we have $a \in A$ with $a^*a \in B$ but $aa^* \notin B$, and we may set $u = u_a$, $v = v_a$, $C = aAa^*$ and $D = vA_av^*$. Then $C \subseteq \overline{a^*Aa} \vee D \subseteq B \vee D$ and hence $C \subseteq (B \vee C) \wedge (B \vee D)$, even though $C \cap D = \{0\}$ and hence $C \not\subseteq B = B \vee (C \wedge D)$. \square

But, unlike with \vee -distributivity, even ideals in $\mathcal{H}(A)$ can fail to be \wedge -distributive.

Example 3. Let $A = C(\mathbb{N} \cup \{\infty\}, M_2)$ and identify A'' with all bounded functions from $\mathbb{N} \cup \{\infty\}$ to M_2 . Define $p \in \mathcal{P}(A' \cap A'')^\circ$ by $p(n) = n \bmod 2$ and $p(\infty) = 0$. Also define $q, r \in \mathcal{P}(A' \cap A'')^\circ$ by $q(n) = P_{1/n}$ and $r(n) = P_{(-1)^n/n}$ (and $q(\infty) = P_0 = r(\infty)$) where

$$P_\theta = \begin{bmatrix} \sin \theta \\ \cos \theta \end{bmatrix} \begin{bmatrix} \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \sin^2 \theta & \sin \theta \cos \theta \\ \sin \theta \cos \theta & \cos^2 \theta \end{bmatrix} \in \mathcal{P}(M_2).$$

Then $(q \wedge r)(n) = (n \bmod 2)P_{1/n}$, for all $n \in \mathbb{N}$, and hence $(q \wedge r)^\circ(\infty) = 0$ so $(p \vee (q \wedge r)^\circ)(\infty) = 0$. But $(p \vee q)(n) = 1 - (n \bmod 2)P_{1/n+\pi/2} = (p \vee r)(n)$, for all $n \in \mathbb{N}$, and $(p \vee q)(\infty) = P_0 = (p \vee r)(\infty)$, so $p \vee q = ((p \vee q) \wedge (p \vee r))^\circ = p \vee r$ and hence

$$((p \vee q) \wedge (p \vee r))^\circ(\infty) = P_0 \neq 0 = (p \vee (q \wedge r)^\circ)(\infty).$$

This is somewhat surprising, given that when every $p \in \mathbb{P}$ is \vee -distributive, every $p \in \mathbb{P}$ is also \wedge -distributive (see e.g. [Bly05] §5.1). In this case we simply call \mathbb{P} *distributive*. The $B \in \mathcal{H}(A)$ for which $\mathcal{H}(B)$ is distributive can be characterized in several different ways, as we now show (see also [PZ00] Lemma 2.6).

We denote the closed ideals of A by $\mathcal{I}(A)$ and call $B \in \mathcal{H}(A)$ *ideal-finite* if $\overline{\text{span}}(ACA) = \overline{\text{span}}(ABA)$ implies $C = B$, for all $C \in \mathcal{H}(B)$.

Corollary 4. *For any $B \in \mathcal{H}(A)$, the following are equivalent.*

- (1) B is commutative.
- (2) $\mathcal{H}(B) = \mathcal{I}(B)$.
- (3) B is ideal-finite.
- (4) $\mathcal{H}(B)$ is distributive.

Proof.

- (1) \Rightarrow (2) If B is commutative and $C \in \mathcal{H}(B)$ then $BC = CB = BC \cap CB \subseteq C$.
- (2) \Rightarrow (3) If $C \in \mathcal{I}(B)$ then $C = B \cap \overline{\text{span}}(ACA)$, so if $\overline{\text{span}}(ACA) = \overline{\text{span}}(ABA)$ then $C = B \cap \overline{\text{span}}(ABA) = B$.
- (2) \Rightarrow (4) $C, D, E \in \mathcal{I}(B) \Rightarrow C \wedge (D \vee E) = \overline{CD + CE} = \overline{CD} + \overline{CE} = (C \wedge D) \vee (C \wedge E)$.
- (4) \Rightarrow (2) See Theorem 4.

- (3) \Rightarrow (1) If B is not commutative then $\text{rank}(\pi(p_B)) > 1$, for some $\pi \in \hat{A}$. Taking $v \in \mathcal{R}(\pi(p_B)) \setminus \{0\}$, we have $C = \{b \in B : \pi(b)v = 0 = \pi(b^*)v\} \subsetneq B$, even though $\overline{\text{span}}(ACA) = \overline{\text{span}}(ABA)$.

□

Question 5. *Are the \wedge -distributive elements of $\mathcal{H}(A)$ precisely those of the form $I \oplus pAp$ where $I \in \mathcal{I}(A)$ is commutative and $p \in \mathcal{P}(\mathcal{M}(A)) \cap A'$?*

While we do not know the answer to this question, or even if there is any algebraic characterization of \wedge -distributivity in $\mathcal{H}(A)$, we can obtain a number of characterizations of \wedge -distributivity. First, let us introduce the order theoretic notion of centrality.

Note that, for any p and q in a poset \mathbb{P} , $[p, q]$ denotes the interval they define, i.e. $[p, q] = \{r \in \mathbb{P} : p \leq r \leq q\}$.

Definition 11. We call $p \in \mathbb{P}$ *central* if $\mathbb{P} \cong [0, p] \times [0, p^\perp]$, for some $p^\perp \in \mathbb{P}$, via

$$(q, r) \mapsto q \vee r \quad \text{and} \quad s \mapsto (s \wedge p, s \wedge p^\perp).$$

As with complements, the central elements of $\mathcal{P}(X)^\circ$ are precisely the clopen sets. However, just like with \wedge -pseudocomplements, central elements in $\mathcal{H}(A)$ must be ideals. In fact, for $p \in \mathcal{P}(A'')^\circ$, the algebraic notion of central, in $\mathcal{M}(A)$, coincides with the order theoretic notion just defined, as well as to the dual of several other order theoretic notions examined so far.

Theorem 6. *For any p in $\mathcal{P}(A'')^\circ$, the following are equivalent.*

- (1) $A = A_p \oplus A_{p^\perp}$.
- (2) $p \in \mathcal{M}(A) \cap A'$.
- (3) p is central.
- (4) p is \wedge -distributive.
- (5) p is/has a \wedge -distributive complement.
- (6) p is/has a \vee -pseudocomplement.

Proof. We immediately see that (3) \Rightarrow p is a \vee -pseudocomplement and

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow p \text{ has a } \vee\text{-pseudocomplement.}$$

Also Theorem 1 (8) \Rightarrow (5) yields (6) \Rightarrow (1) here. Another immediate implication is (3) \Rightarrow (5), while Theorem 5 and Theorem 1 (3) \Rightarrow (5) yields (5) \Rightarrow (1) here. □

Before moving on, let us point out that the centre of $\mathcal{H}(A)$ may not be complete, even though $\mathcal{H}(A)$ itself is a complete lattice. Indeed, the p in Example 3 is not in A , even though it is a (countable) supremum and infimum (taken in $\mathcal{P}(A'')^\circ$) of projections in $A \cap A'$. Whether this could happen in any complete lattice was mentioned as a question of S. Holland in [Bir67] Ch 5 Problem 34, with examples and related theory given in [Jak73] and [Jan78]. Example 3 shows that this situation crops up quite naturally in $\mathcal{H}(A)$, even for quite elementary C*-algebras A .

8. SEPARATIVITY AND *-ANNIHILATORS

Dual to subfiniteness, we have separativity, i.e. $p \in \mathbb{P}$ is *separative* if p is \wedge -separated from every $q \not\leq p$. We also call \mathbb{P} *separative* (see [Kun80]) when every $p \in \mathbb{P}$ is separative, although some authors would call such \mathbb{P} SSC. Instead, we work with the following slightly weaker definition of SSC (which agrees with the original definition in [MM70])

Definition 12. We call p *section \wedge -semicomplemented* or *SSC* if p is \wedge -separated from every $q > p$. We call \mathbb{P} itself SSC when every $p \in \mathbb{P}$ is SSC.

Yet again, let us consider the lattice of open sets $\mathcal{P}(X)^\circ$ of a topological space X . If $O \in \mathcal{P}(X)^\circ$ is regular then, for any $N \in \mathcal{P}(X)^\circ$ with $N \not\subseteq O$, we must also have $N \not\subseteq \overline{O}$ (because $N \subseteq \overline{O}$ would imply $N = N^\circ \subseteq \overline{O}^\circ = O$) and hence $N \setminus \overline{O}$ is a non-empty \wedge -semicomplement of O in $\mathcal{P}(X)^\circ$. While if O is not regular then $O \subsetneq \overline{O}^\circ$, even though the definition of closure means there is no non-empty open $N \subseteq \overline{O}$ with $O \cap N = \emptyset$. So, as with \wedge -pseudocomplements, the SSC/separative elements of $\mathcal{P}(X)^\circ$ are precisely the regular open subsets. However, unlike \wedge -pseudocomplements, SSC/separative elements of $\mathcal{H}(A)$ need not be ideals, which naturally leads to the following question – are the SSC/separative elements of $\mathcal{H}(A)$ precisely the $*$ -annihilators?

This time, the answer is no in general (see [Example 4](#) below). However, we can prove one direction, namely that $*$ -annihilators in $\mathcal{H}(A)$ are necessarily separative, and in fact satisfy a strong version of being SSC too. For this, we first require a number of slightly technical spectral projection inequalities.⁹

Lemma 3. *For $\epsilon, \lambda > 0$, there exists $\delta > 0$ such that, whenever $b, c \in \mathcal{B}(H)_+^1$, $c \leq q \in \mathcal{P}(\mathcal{B}(H))$ and $\|bq\|^2 \leq \lambda + \delta$, we have*

$$(8.1) \quad \|c_{[0,1-\epsilon]}(cb^2c)_{[\lambda-\delta,1]}\| \leq \epsilon.$$

Proof. If $\epsilon \geq 1$ then (8.1) holds trivially, so assume $\epsilon < 1$. For any $v \in H$,

$$\begin{aligned} \|cv\|^2 &= \|cc_{[0,1-\epsilon]}v\|^2 + \|cc_{(1-\epsilon,1]}v\|^2 \\ &\leq (1-\epsilon)^2 \|c_{[0,1-\epsilon]}v\|^2 + \|c_{(1-\epsilon,1]}v\|^2 \\ &\leq \|v\|^2 - \epsilon(2-\epsilon) \|c_{[0,1-\epsilon]}v\|^2 \\ (8.2) \quad &\leq \|v\|^2 - \epsilon \|c_{[0,1-\epsilon]}v\|^2 \quad (\text{as } \epsilon \leq 1). \end{aligned}$$

As $c \leq q$, we have $q^\perp cq^\perp \leq q^\perp qq^\perp = 0$ and hence $q^\perp c = 0$, i.e. $c = qc$. Thus, for $v \in \mathcal{R}((cb^2c)_{[\lambda-\delta,1]})$,

$$\begin{aligned} (\lambda - \delta) \|v\|^2 &\leq \langle cb^2cv, v \rangle \\ &= \|bcv\|^2 \\ &\leq \|bq\|^2 \|cv\|^2, \text{ as } c = qc, \\ &\leq (\lambda + \delta) (\|v\|^2 - \epsilon \|c_{[0,1-\epsilon]}v\|^2), \text{ by (8.2), so} \\ (\lambda + \delta) \epsilon \|c_{[0,1-\epsilon]}v\|^2 &\leq 2\delta \|v\|^2 \text{ and} \\ \|c_{[0,1-\epsilon]}v\|^2 &\leq 2\delta \|v\|^2 / (\lambda\epsilon), \end{aligned}$$

which immediately yields (8.1), for $\delta \leq \lambda\epsilon^3/2$. \square

Lemma 4. *For $\epsilon, \lambda > 0$, there exists $\delta > 0$ such that, whenever $b, c \in \mathcal{B}(H)_+^1$, $c \leq q \in \mathcal{P}(\mathcal{B}(H))$ and $\|bq\|^2 \leq \lambda + \delta$, we have*

$$(8.3) \quad \|(1-c)(cb^2c)_{[\lambda-\delta,1]}\| \leq \epsilon.$$

⁹Many of the results that follow first appeared in the preprint [\[Bic13a\]](#).

Proof. Replacing ϵ with $\epsilon/\sqrt{2}$ in Lemma 3, we obtain $\delta > 0$ such that, for any $v \in \mathcal{R}((cb^2c)_{[\lambda-\delta,1]})$,

$$\begin{aligned} \|(1-c)v\|^2 &= \|(1-c)c_{[0,1-\epsilon/\sqrt{2}]}v\|^2 + \|(1-c)c_{[1-\epsilon/\sqrt{2},1]}v\|^2 \\ &\leq \|c_{[0,1-\epsilon/\sqrt{2}]}v\|^2 + \epsilon^2\|c_{[1-\epsilon/\sqrt{2},1]}v\|^2/2 \\ &\leq \epsilon^2\|v\|^2/2 + \epsilon^2\|v\|^2/2, \text{ by (8.1).} \end{aligned}$$

□

The following result generalizes [Bic12] Lemma 5.3.

Lemma 5. *For $\epsilon, \lambda > 0$, there exists $\delta > 0$ such that, whenever $b, c \in \mathcal{B}(H)_+^1$, $c \leq q \in \mathcal{P}(\mathcal{B}(H))$ and $\|bq\|^2 \leq \lambda + \delta$, we have*

$$(8.4) \quad \|b_{[0,\sqrt{\delta}]}(cb^2c)_{[\lambda-\delta,1]}\|^2 \leq 1 - \lambda + \epsilon.$$

Proof. Let $\delta > 0$ be that obtained in Lemma 4 from replacing ϵ with $\epsilon/4$. If necessary, replace δ with a smaller non-zero number so that we also have

$$(8.5) \quad (1 - \lambda + \delta + \epsilon/2)/(1 - \delta) \leq 1 - \lambda + \epsilon.$$

Then, for all $v \in \mathcal{R}((cb^2c)_{[\lambda-\delta,1]})$,

$$\begin{aligned} (\lambda - \delta)\|v\|^2 &\leq \langle cb^2cv, v \rangle \\ &= \langle b^2cv, cv \rangle \\ &\leq \langle b^2v, v \rangle + \epsilon\|v\|^2/2, \text{ by (8.3),} \\ &= \langle b^2b_{[0,\sqrt{\delta}]}v, v \rangle + \langle b^2b_{[\sqrt{\delta},1]}v, v \rangle + \epsilon\|v\|^2/2 \\ &\leq \delta\langle b_{[0,\sqrt{\delta}]}v, v \rangle + \langle b_{[\sqrt{\delta},1]}v, v \rangle + \epsilon\|v\|^2/2 \\ &= \delta\|b_{[0,\sqrt{\delta}]}v\|^2 + (\|v\|^2 - \|b_{[0,\sqrt{\delta}]}v\|^2) + \epsilon\|v\|^2/2, \text{ so} \\ (1 - \delta)\|b_{[0,\sqrt{\delta}]}v\|^2 &\leq (1 - \lambda + \delta + \epsilon/2)\|v\|^2, \text{ and hence} \\ \|b_{[0,\sqrt{\delta}]}v\|^2 &\leq (1 - \lambda + \epsilon)\|v\|^2, \text{ by (8.5).} \end{aligned}$$

□

Definition 13. For $\epsilon \in (0, 1]$, we call $B \in \mathcal{H}(A)$ ϵ -SSC if, whenever $B \subsetneq C \in \mathcal{H}(A)$, there exists $D \in \mathcal{H}(C)$ with $\|p_B p_D\| < \epsilon$.

So the smaller ϵ is, the stronger the ϵ -SSC property is, and if $B \in \mathcal{H}(A)$ is even 1-SSC then it is SSC, according to Definition 12. So the following result answers one direction of our original question.

Theorem 7. *Any $B \in \mathcal{H}(A)$ with $B = B^{\perp\perp}$ is 1-SSC.*

Proof. Take $C \in \mathcal{H}(A)$ with $B \subsetneq C$, so we have $c \in C_+^1 \setminus B$. This means we have $b \in B_+^{\perp\perp}$ with $bc \neq 0$, and hence $bq \neq 0$, where $q = c_{(0,1]}$. Set $\lambda = \|bq\|^2$, take positive $\epsilon < \lambda$ and let $\delta > 0$ be that obtained in Lemma 5. Note that we may now assume that $\|bc\|^2 > \lambda - \delta$ by replacing c with $f(c)$, where f is a continuous function on $[0, 1]$ with $[\mu, 1] \subseteq f^{-1}\{1\}$, for sufficiently small $\mu > 0$. Set $p = (cb^2c)_{(\lambda-\delta,1]} \in \mathcal{P}(C''')^\circ \setminus \{0\}$. As $p_B \leq b_{\{0\}} \leq b_{[0,\sqrt{\delta}]}$, Lemma 5 yields

$$\|p_B p\|^2 \leq \|b_{[0,\sqrt{\delta}]}(cb^2c)_{(\lambda-\delta,1]}\|^2 \leq 1 - \lambda + \epsilon < 1.$$

□

In fact, we can do much better than 1-SSC, but first we need some more results.

Lemma 6. *For $\epsilon, \lambda > 0$, there exists $\delta > 0$ such that, whenever $b, c \in \mathcal{B}(H)_+^1$, $p, q \in \mathcal{P}(\mathcal{B}(H))$, $b \leq p$, $c \leq q$ and $\|pq\|^2 \leq \lambda + \delta$, we have*

$$(8.6) \quad \|p(cb^2c)_{[\lambda-\delta, 1]}\|^2 \leq \lambda + \epsilon.$$

Proof. Let $\delta > 0$ be that obtained in Lemma 4 with ϵ replaced with $\epsilon/4$. If necessary, decrease δ so that $\delta \leq \epsilon/2$. Then, for $v \in \mathcal{R}((cb^2c)_{[\lambda-\delta, 1]})$,

$$\begin{aligned} \|pv\|^2 &= \langle pv, pv \rangle \\ &\leq \langle pcv, pcv \rangle + \epsilon \|v\|^2/2 \\ &\leq (\|pc\|^2 + \epsilon/2) \|v\|^2 \\ &\leq (\lambda + \delta + \epsilon/2) \|v\|^2. \\ &\leq (\lambda + \epsilon) \|v\|^2. \end{aligned}$$

□

For use in the next result, note that whenever $p, q \in \mathcal{P}(A)$ and $p \neq 0$,

$$(8.7) \quad \|pq\|^2 + \|pq^\perp\|^2 \geq 1.$$

For simply take $v \in \mathcal{R}(p) \setminus \{0\}$ and note that

$$\|v\|^2 = \|qv\|^2 + \|q^\perp v\|^2 = \|qp v\|^2 + \|q^\perp p v\|^2 \leq (\|qp\|^2 + \|q^\perp p\|^2) \|v\|^2.$$

Also note that

$$(8.8) \quad \|pq^\perp\|^2 \leq \lambda \iff pq^\perp p \leq \lambda p \iff (1 - \lambda)p \leq pqp.$$

In fact, the following result is a natural modulo- ϵ generalization of (8.7) from $\mathcal{P}(A)$ to $\mathcal{H}(A)$, where B , C and D correspond to q , p and q^\perp respectively.

Theorem 8. *For $\epsilon > 0$ and $B, C \in \mathcal{H}(A) \setminus \{0\}$ with $\|p_B p_C\|^2 = \lambda < 1$, there exists $D \in \mathcal{H}(A)$ with $\|p_B p_D\| \leq \epsilon$ and $\|p_C p_D\|^2 \geq 1 - \lambda - \epsilon$.*

Proof. Choose $\delta > 0$ small enough that it satisfies Lemma 5 and Lemma 6 with ϵ replaced by some $\mu > 0$, to be determined later. Take $c \in C_+^1$ and $b \in B_+^1$ with $\|bc\|^2 > \lambda - \delta/2$. Take $c' \in C_+^1$ with $(cb^2c)_{[\lambda-\delta/2, 1]} \leq c' \leq (cb^2c)_{[\lambda-\delta, 1]}$ and let $a = (1 - f(b))c'^2(1 - f(b))$, where f is continuous, 0 on $[0, \delta/2]$ and 1 on $[\delta, 1]$, so

$$\begin{aligned} \|a\| &= \|(1 - f(b))c'\|^2 \\ &\geq \|b_{\{0\}}(cb^2c)_{[\lambda-\delta/2, 1]}\|^2 \\ &\geq 1 - \|b_{(0, 1]}(cb^2c)_{[\lambda-\delta, 1]}\|^2, \text{ by (8.7)} \\ (8.9) \quad &\geq 1 - \lambda - \mu, \text{ by (8.6).} \end{aligned}$$

In particular, $\|a\| > 0$ as long as $\mu < 1 - \lambda$, and we may define $a' = \|a\|^{-1}a$.

By (8.4), we have

$$(8.10) \quad \|b_{[0, \sqrt{\delta}]}c'_{(0, 1]}\|^2 \leq \|b_{[0, \sqrt{\delta}]}(cb^2c)_{[\lambda-\delta, 1]}\|^2 \leq 1 - \lambda + \mu$$

and so, by (8.8),

$$(8.11) \quad c'_{(0, 1]}b_{[\delta, 1]}c'_{(0, 1]} \geq c'_{(0, 1]}b_{[\sqrt{\delta}, 1]}c'_{(0, 1]} \geq (\lambda - \mu)c'_{(0, 1]}.$$

Thus

$$\begin{aligned}
(1 - \lambda - \mu) \|p_B a' p_B\| &\leq \|a\| \|p_B a' p_B\|, \text{ by (8.9)} \\
&= \|p_B a p_B\| \\
&= \|p_B (1 - f(b)) c'^2 (1 - f(b)) p_B\| \\
&= \|(p_B - f(b)) c'\|^2 \\
&= \|c' (p_B - f(b))^2 c'\| \\
&\leq \|c' (p_B - f(b)) c'\| \\
&= \|c' (p_C p_B p_C - c'_{(0,1]} b_{[\delta,1]} c'_{(0,1]}) c'\| \\
&\leq \|c' (\lambda - \lambda + \mu) c'\|, \text{ by (8.11) and } \|p_B p_C\|^2 = \lambda \\
&\leq \mu, \text{ and hence,} \\
\|p_B a' p_B\| &\leq \mu / (1 - \lambda - \mu).
\end{aligned}$$

Set $p = a'_{(1-\mu,1]} \in \mathcal{P}(A'')^\circ$ and $D = A_p$. Now

$$\|p_B p\|^2 = \|p_B p p_B\| \leq \|p_B a' p_B\| / (1 - \mu) \leq \mu / ((1 - \lambda - \mu)(1 - \mu))$$

so, as long as $\mu > 0$ was chosen sufficiently small, $\|p_B p\| \leq \epsilon$.

Also, $\|(1 - f(b))_{(0,1]} c'\|^2 \leq \|b_{[0,\sqrt{\delta}]} c'_{(0,1]}\|^2 \leq 1 - \lambda + \mu$, by (8.10). This means, as long as we chose μ at least half as small as the δ obtained in Lemma 5 (from the given ϵ) we can apply (8.4) with b, c and λ replaced by $c', 1 - f(b)$ and $1 - \lambda$ to get

$$\|c'_{[0,\sqrt{\mu}]} a'_{(1-\mu,1]}\|^2 \leq \|c'_{[0,\sqrt{2\mu}]} ((1 - f_\delta(b)) c'^2 (1 - f_\delta(b)))_{(1-\lambda-2\mu,1]}\|^2 \leq \lambda + \epsilon,$$

(the first inequality follows from (8.9) and the fact $(1 - \mu)(1 - \lambda - \mu) \geq 1 - \lambda - 2\mu$). Thus $\|p_C p\|^2 \geq \|c'_{[\sqrt{\mu},1]} a'_{(1-\mu,1]}\|^2 \geq 1 - \lambda - \epsilon$. \square

For any 1-SSC $B \in \mathcal{H}(A)$, the first part of the above result can be applied within any hereditary C*-subalgebra containing B to show that B must actually be ϵ -SSC, for any $\epsilon > 0$. So Theorem 7 can immediately be strengthened as follows.

Corollary 5. *Any $B \in \mathcal{H}(A)$ with $B = B^{\perp\perp}$ is ϵ -SSC, for all $\epsilon > 0$.*

It is natural to wonder if this can be strengthened just a little more to bring ϵ down to 0, i.e. to show that whenever $B, C \in \mathcal{H}(A)$, $B = B^{\perp\perp}$ and $B \subsetneq C$, we have $C \cap B^\perp \neq \{0\}$ (such a B might well be called *section \perp -semicomplemented* or *orthomodular*). The following example shows that this is not possible in general.

Example 4. Let $A = C([0,1], \mathcal{K}(H))$, where $\mathcal{K}(H)$ is the C*-algebra of compact operators on a separable infinite dimensional Hilbert space H . Identify A'' (in the atomic representation) with all bounded functions from $[0,1]$ to $\mathcal{B}(H)$. Now let $p_n \in A$ be the rank 1 projection onto $\mathbb{C}e_n$, for each $n \in \mathbb{N}$, where (e_n) is an orthonormal basis for H . Also let (r_n) enumerate a countable dense subset of $(0,1)$ and let χ_S denote the characteristic function of $S \subseteq [0,1]$. Consider $p = \bigvee \chi_{[0,r_n]} p_n \in \mathcal{P}(A'')^\circ$ and $p' = \bigvee \chi_{(r_n,1]} p_n \in \mathcal{P}(A'')^\circ$. For any $a \in A_p^{\perp\perp}$, $p_n a(x) = 0$, for all $x \in [0, r_n)$ which, as a is continuous, means $p_n a(r_n) = 0$ too so $a \leq p'$. Thus $A_{p'} = A_p^\perp$ and, likewise, $A_p = A_{p'}^\perp = A_p^{\perp\perp}$.

Now let q be the (constant) rank one projection onto $v = \sum 2^{-n} e_n$ and consider $p \vee q \in \mathcal{P}(A'')^\circ$. As $q \in A_{p \vee q} \setminus A_p$, we certainly have $A_p \subsetneq A_{p \vee q}$. But $p \vee q - p$ is a rank 1 projection on $(0,1)$ which is discontinuous on the dense subset (r_n) , so $A_{p \vee q} \cap A_p^\perp = A_{p \vee q - p} = \{0\}$.

Furthermore, the A_p above is SSC in $\mathcal{H}(A)$, by [Theorem 7](#), and so certainly SSC in $\mathcal{H}(A_{p \vee q})$. But we just showed that A_p is not a *-annihilator in $A_{p \vee q}$, and thus we can not hope to use the SSC property to characterize *-annihilators in general.

Question 6. *Is there an order theoretic characterization of *-annihilators in $\mathcal{H}(A)$?*

Still considering [Example 4](#) above, note that, as A_p is 1-SSC in $A_{p \vee q}$, we have $b \in B_+^1 \in \mathcal{H}(A_{p \vee q})$ with $\|bp\| \leq \|p_B p\| < 1 = \|b\|$, despite the fact p is dense in $p \vee q$ (equivalently, $(p \vee q - p)^\circ = 0$), i.e. p is *non-regular* in the sense of [\[Tom60\]](#) (this concept of regularity has little to do with the topological regularity of open sets discussed earlier). The question of whether there exist open dense non-regular projections was mentioned as an open problem in [\[PZ00\]](#), and the first examples were given in [\[AE02\]](#) (which inspired our construction of [Example 4](#)). In [\[AE02\]](#), a constant γ was even defined to measure the degree of regularity of an open dense projection p , essentially by

$$\gamma(p) = \inf_{q \in \mathcal{P}(A'')^\circ} \|pq\|,$$

where $\gamma(p) = 1$ means p is regular and lower $\gamma(p)$ values signify lower regularity. However, [Theorem 8](#) shows that if $\gamma(p) < 1$ then, in fact, $\gamma(p) = 0$, i.e. any non-regular open dense projection must actually be as non-regular as possible.

9. THE *-ANNIHILATOR ORTHOLATTICE

We have just seen in the previous section (and [§6](#)) that *-annihilators have special properties within $\mathcal{H}(A)$, and one might guess they could be worthy of study in their own right. Indeed, a surprisingly detailed *-annihilator theory, closely resembling the basic theory of projections in von Neumann algebras, can be developed even in the much broader context of *-semigroups (see [\[Bic14a\]](#)). Here we investigate what more can be said about them in the C*-algebra context, and how closely related the *-annihilator ortholattice $\mathcal{P}(A)^\perp$ is to the hereditary C*-subalgebra lattice $\mathcal{H}(A)$.

Again consider a topological space X , but this time assume it also satisfies the T_3 separation axiom, i.e. any disjoint point and closed subset have disjoint neighbourhoods. Thus, whenever $x \in O \in \mathcal{P}(X)^\circ$, we have $N \in \mathcal{P}(X)^\circ$ with $x \in N$ and $\overline{N} \subseteq O$, and hence $x \in \overline{N}^\circ \subseteq O$. As $x \in O$ was arbitrary,

$$O = \bigcup \{N \subseteq X : N = \overline{N}^\circ \subseteq O\},$$

i.e. the regular open subsets of X are \vee -dense in $\mathcal{P}(X)^\circ$. As any locally compact Hausdorff space is T_3 , it follows that $\mathcal{P}(A)^\perp$ is \vee -dense in $\mathcal{H}(A)$ whenever A is commutative. Yet again, the commutativity assumption here is unnecessary, as we now show.

First note that, for any $a \in A_+^1$ and $\lambda \in (0, 1)$, $a_{[0, \lambda]}$ is open. Indeed, letting f be any continuous function on $[0, 1]$ that is non-zero on $[0, \lambda)$ and 0 on $[\lambda, 1]$, we have $a_{[0, \lambda]} = f(a)_{(0, 1]} = p_{\overline{f(a)Af(a)}}$. The only slight problem is that $f(a) \notin A$ when A is not unital, but we still have $a_{[0, \lambda]} = p_B$ for some $B \in \mathcal{H}(A)$, by [\[Ped79\]](#) Proposition 3.11.9. As $A_{a_{[0, \lambda]}} \subseteq A_{a_{(\lambda, 1]}}^\perp$, we have $A_{a_{(\lambda, 1]}}^{\perp\perp} \subseteq A_{a_{[0, \lambda]}}^\perp \subseteq A_{a_{[\lambda, 1]}}$. In other words, letting $p = p_{A_{a_{(\lambda, 1]}}^{\perp\perp}} \in \mathcal{P}(A'')^\circ$, we have $A_p^{\perp\perp} = A_p$ and

$$(9.1) \quad a_{(\lambda, 1]} \leq p \leq a_{[\lambda, 1]}.$$

Now take any $B \in \mathcal{H}(A)$ and let (f_n) be a sequence of continuous functions on $[0, 1]$ uniformly approaching the identity with $[0, 1/n] \subseteq f_n^{-1}\{0\}$, for all $n \in \mathbb{N}$.

Then, for any $b \in B_+^1$ and $n \in \mathbb{N}$, we have $p \in \mathcal{P}(A'')^\circ$ with $A_p = A_p^{\perp\perp}$ and $b_{(1/n,1]} \leq p \leq b_{[1/n,1]}$ and hence $f_n(b) \in A_p \subseteq \overline{bAb} \subseteq B$. As $f_n(b) \rightarrow b$, we have

$$b \in B_\vee = \bigvee (\mathcal{P}(A)^\perp \cap \mathcal{H}(B)) = \bigvee \{C \in \mathcal{H}(B) : C^{\perp\perp} = C\}.$$

As b was arbitrary, $B = B_\vee$. As B was arbitrary, $\mathcal{P}(A)^\perp$ is \vee -dense in $\mathcal{H}(A)$.

Any complete lattice \mathbb{Q} that is \vee -dense in another poset \mathbb{P} must in fact be a complete \wedge -sublattice of \mathbb{P} , i.e. infimums in \mathbb{Q} are also valid in \mathbb{P} . For if q is the infimum of S in \mathbb{Q} and $p \in \mathbb{P}$ satisfies $p \leq s$, for all $s \in S$, then

$$p = \bigvee \{r \in \mathbb{Q} : r \leq p\} \leq \bigvee \{r \in \mathbb{Q} : \forall s \in S (r \leq s)\} = q.$$

In particular, $\mathcal{P}(A)^\perp$ is a complete \wedge -sublattice of $\mathcal{H}(A)$, although this can also be seen directly from $\bigcap_\alpha (B_\alpha^\perp) = (\bigcup_\alpha B_\alpha)^\perp$. However, it is important to note that $\mathcal{P}(A)^\perp$ is not a \vee -sublattice of $\mathcal{H}(A)$, even for commutative A . For example, $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$ are regular open subsets of $[0, 1]$ even though $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ is not.

Proposition 9. *Assume \mathbb{Q} is a \vee -dense \wedge -sublattice of SSC elements in \mathbb{P} . Then every $q \in \mathbb{Q}$ is separative in both \mathbb{P} and \mathbb{Q} .*

Proof. Take $q \in \mathbb{Q} \setminus \{0\}$ and $p \in \mathbb{P} \setminus \{0\}$ with $p \not\leq q$. As \mathbb{Q} is \vee -dense in \mathbb{P} , we have $r \in \mathbb{Q} \setminus \{0\}$ with $r \leq p$ and $r \not\leq q$, and hence $q \wedge r < r$. But \mathbb{Q} is a \wedge -sublattice so $q \wedge r \in \mathbb{Q}$. As elements of \mathbb{Q} are SSC in \mathbb{P} , we have $s \in \mathbb{P} \setminus \{0\}$ with $s \leq r \leq p$ and $s \wedge q = s \wedge q \wedge r = 0$. Thus q is separative in \mathbb{P} and, again using join density (actually order density would be enough), we can replace s with an element of \mathbb{Q} to show that q is separative in \mathbb{Q} too. \square

In particular, any SSC \wedge -lattice is separative. Also, [Theorem 7](#) now immediately yields the following.

Corollary 6. *Every $B \in \mathcal{P}(A)^\perp$ is separative in both $\mathcal{H}(A)$ and $\mathcal{P}(A)^\perp$.*

Next we examine the elements of $\mathcal{P}(A)^\perp$ with special order properties, as in the previous sections. Note that, as $\mathcal{P}(A)^\perp$ is not a \vee -sublattice of $\mathcal{H}(A)$, various lattice theoretic concepts can potentially have very different meanings in $\mathcal{P}(A)^\perp$ and $\mathcal{H}(A)$. For example, complements in $\mathcal{H}(A)$ are quite special, while every $B \in \mathcal{P}(A)^\perp$ has a complement in $\mathcal{P}(A)^\perp$, namely B^\perp . In fact, $^\perp$ is an *orthocomplementation* on $\mathcal{P}(A)^\perp$ (while it is merely a *Galois \wedge -semicomplementation* on $\mathcal{H}(A)$), which makes $\mathcal{P}(A)^\perp$ an *ortholattice*, giving us access to ortholattice theory and concepts, like the following.

Definition 14. ¹⁰ In an ortholattice \mathbb{P} , we define the *commutativity* relation C by

$$pCq \iff p = (p \wedge q) \vee (p \wedge q^\perp).$$

We also define the *Elkan* relation E by

$$pEq \iff p \vee q = (p \wedge q^\perp) \vee q.$$

Proposition 10. *For q in a separative ortholattice \mathbb{P} , the following are equivalent.*

- (1) q is central.
- (2) q is \vee -distributive.

¹⁰For more information on the commutativity relation C , particularly in orthomodular lattices, see [\[Kal83\]](#) or [\[Ber85\]](#). On the other hand, the Elkan relation E does not seem to have been formally defined before, although a similar global condition, *Elkan's Law*, was studied in [\[KD08\]](#).

- (3) pCq , for all $p \in \mathbb{P}$.
- (4) pEq , for all $p \in \mathbb{P}$.
- (5) q is a \wedge -pseudocomplement of q^\perp .

Proof. Even without separativity, we immediately see that

$$(1) \Rightarrow (2) \text{ or } (3) \Rightarrow (4) \Rightarrow (5),$$

and (3) \Rightarrow (1), by [Mac64] Theorem 3.2. While if (3) fails then, for some $p \in \mathbb{P}$, we have $(p \wedge q) \vee (p \wedge q^\perp) < p$. If \mathbb{P} is separative/SSC, then we have non-zero $r \leq p$ with $r \wedge q = r \wedge p \wedge q \leq r \wedge ((p \wedge q) \vee (p \wedge q^\perp)) = 0$. Likewise, $r \wedge q^\perp = 0$, which means q is not a \wedge -pseudocomplement of q^\perp , proving (5) \Rightarrow (3) (a similar argument appears in the proof of [MM70] Theorem (4.18)). \square

Theorem 9. For any $B \in \mathcal{P}(A)^\perp$, the following are equivalent in $\mathcal{P}(A)^\perp$.

- (1) B is an ideal.
- (2) B is central.
- (3) B is \vee -distributive.
- (4) B is/has a \wedge -pseudocomplement.
- (5) B is/has a unique complement.

Proof. (2) \Rightarrow (3),(4),(5) is immediate. The other implications are proved as follows.

(1) \Rightarrow (2) See [Bic14a] Corollary 5.2.

(2) \Rightarrow (1) If $B \in \mathcal{P}(A)^\perp$ is not an ideal then $B^{\perp\perp} = B \subsetneq ABA$ and thus we have $a \in A \setminus \{0\}$ with $a^*a \in B$ and $aa^* \in B^\perp$. Define $u = u_a$ and $v = v_a$ which, as $\mathcal{P}(A)^\perp$ is \leq -dense, means we have $C \in \mathcal{P}(A)^\perp \setminus \{0\}$ with $C \subseteq vA_a v^*$. As $C \subseteq A_a$,

$$B^\perp \cap C = B^\perp \cap A_a \cap C = \overline{aAa^*} \cap C \subseteq \overline{aAa^*} \cap vA_a v^*.$$

But, as shown in the proof of Lemma 1, $vv^*uu^*vv^* = \frac{1}{2}vv^*$ so $\|vv^*uu^*\| = \frac{1}{\sqrt{2}} < 1$ and hence $vA_a v^* \cap \overline{aAa^*} = \{0\}$. Likewise

$$B \cap C = B \cap A_a \cap C = \overline{a^*Aa} \cap C \subseteq \overline{a^*Aa} \cap vA_a v^* = \{0\},$$

so $C \neq \{0\} = (C \cap B) \vee (C \cap B^\perp)$ and hence B is not central.

(3) \Rightarrow (2) If B is \vee -distributive then B^\perp is \vee -distributive and hence central, by Proposition 10, so B is central too.

(4) \Rightarrow (1) Say B is a \wedge -pseudocomplement of C in $\mathcal{P}(A)^\perp$. As $\mathcal{P}(A)^\perp$ is a \vee -dense \wedge -sublattice of $\mathcal{H}(A)$, B must also be a \wedge -pseudocomplement of C in $\mathcal{H}(A)$ and hence $B = C^\perp$ is an ideal, by Theorem 3. Likewise, if C is a \wedge -pseudocomplement of B in $\mathcal{P}(A)^\perp$ then $C = B^\perp$ is an ideal, as is $B = B^{\perp\perp}$.

(5) \Rightarrow (1) If $B = B^{\perp\perp}$ is not an ideal, then neither is B^\perp and hence there exists $a \in A_+^1$ that does not commute with p_{B^\perp} . Then, as in the proof of Corollary 1, we have a unitary $u \in \mathcal{M}(A)$ with $0 < \|p_{B^\perp} - u^*p_{B^\perp}u\| < 1$, and hence $C = u^*B^\perp u$ is another complement of B in $\mathcal{P}(A)^\perp$. So if B has a unique complement, which must be B^\perp , then B is an ideal. Likewise, if B is not an ideal then there exists $a \in A_+^1$ that does not commute with p_B , which allows us to find another complement of B^\perp in $\mathcal{P}(A)^\perp$. So if B is the unique complement of C in $\mathcal{P}(A)^\perp$ then $B = C^\perp$ so $C = B^\perp$ and hence B is an ideal. \square

A purely order theoretic proof of (5) \Rightarrow (2) above would be possible (see [MM70] Theorem (4.20)) if we could show that $\mathcal{P}(A)^\perp$ is not only SSC, but actually SC.

Definition 15. \mathbb{P} is *section complemented (SC)* if $[0, p]$ is complemented, for $p \in \mathbb{P}$.

Question 7. Is $\mathcal{P}(A)^\perp$ section complemented?

We also have the following analog of Corollary 4. As in [Bic14a] Definition 5.5, call $B \in \mathcal{P}(A)^\perp$ ∇ -finite when $C \subseteq B$ and $C^\nabla = B^\nabla$ implies $C = B$, for all $C \in \mathcal{P}(A)^\perp$. We also define $C^{\perp_B} = C^\perp \cap B$ and $C^{\nabla_B} = C^\nabla \cap B$.

Corollary 7. For any $B \in \mathcal{P}(A)^\perp$, the following are equivalent.

- (1) B is commutative.
- (2) $\mathcal{P}(B)^{\perp_B} = \mathcal{P}(B)^{\nabla_B}$.
- (3) B is ∇ -finite.
- (4) $\mathcal{P}(A)^{\perp_B}$ is distributive.
- (5) $B = C^{\perp\perp}$ for some commutative $C \in \mathcal{H}(A)$.

Moreover, if any/all of these conditions is satisfied then $\mathcal{P}(A)^{\perp_B} = \mathcal{P}(B)^{\perp_B}$.

Proof. If B is commutative then $\mathcal{P}(A)^{\perp_B} = \mathcal{P}(B)^{\perp_B}$, by [Bic14a] Theorem 5.4.

- (1) \Rightarrow (2) See Corollary 4 (1) \Rightarrow (2) and [Bic14a] Theorem 4.6.
- (2) \Rightarrow (3) See Corollary 4 (2) \Rightarrow (3) and [Bic14a] Theorem 5.3.
- (1) \Rightarrow (4) Use $\mathcal{P}(A)^{\perp_B} = \mathcal{P}(B)^{\perp_B}$, (1) \Rightarrow (2) and [Bic14a] Corollary 5.2.
- (4) \Rightarrow (2) If $\mathcal{P}(B)^{\perp_B} \neq \mathcal{P}(B)^{\nabla_B}$ then, as in Theorem 9, take $C \in \mathcal{P}(B)^{\perp_B} \setminus \mathcal{P}(B)^{\nabla_B}$ and $D \in \mathcal{P}(B)^{\perp_B} \setminus \{0\}$ such that $C \cap D = \{0\} = C^{\perp_B} \cap D$ and D is in the hereditary C*-subalgebra generated by C and C^{\perp_B} , so $D \subseteq C \vee C^{\perp_B}$ (with the supremum taken in $\mathcal{P}(A)^\perp$ – as we may have $\mathcal{P}(A)^{\perp_B} \neq \mathcal{P}(B)^{\perp_B}$, we may have $C \vee C^{\perp_B} < B$ so this does not follow automatically from $D \subseteq B$). Thus $D \cap (C \vee C^{\perp_B}) = D \neq \{0\} = (D \cap C) \vee (D \cap C^{\perp_B})$ so $\mathcal{P}(A)^{\perp_B}$ is not distributive.
- (3) \Rightarrow (2) If $\mathcal{P}(B)^{\perp_B} \neq \mathcal{P}(B)^{\nabla_B}$ then, taking $C \in \mathcal{P}(B)^{\perp_B} \setminus \mathcal{P}(B)^{\nabla_B}$, we have $C \subsetneq C^{\nabla_B \nabla_B}$ and hence $C \vee C^{\nabla_B} \subsetneq B$, as $C^{\nabla_B \nabla_B}$ is central in $\mathcal{P}(B)^{\perp_B}$, even though $(C \vee C^{\nabla_B})^{\nabla_B \nabla_B} = (C^{\nabla_B} \cap C^{\nabla_B \nabla_B})^{\nabla_B} = \{0\}^{\nabla_B} = B$. By [Bic14a] Theorem 5.3, $(C \vee C^{\nabla_B})^{\nabla_B \nabla_B} = (C \vee C^{\nabla_B})^{\nabla \nabla} \cap B$ so $B \subseteq (C \vee C^{\nabla_B})^{\nabla \nabla}$ and hence, as $C \vee C^{\nabla_B} \subseteq B$, we have $(C \vee C^{\nabla_B})^{\nabla \nabla} = B^{\nabla \nabla}$ so B is not ∇ -finite.
- (2) \Rightarrow (1) By (9.1), any $b \in B$ can be approximated arbitrarily closely by linear combinations of open projections corresponding to *-annihilators of B . If $\mathcal{P}(B)^{\perp_B} = \mathcal{P}(B)^{\nabla_B}$ then each one of these projections is in B' so $B \subseteq B'$.
- (1) \Rightarrow (5) Immediate.
- (5) \Rightarrow (1) We first claim $B \subseteq C'$. If not, we would have $c \in C_+^1$ and $b \in B_+$ such that $bc \neq cb$. Then, for some $\epsilon > 0$, we must have $bc_{[\epsilon, 1]} \neq c_{[\epsilon, 1]}b$ and hence $c_{[\epsilon, 1]}b(1 - c_{[\epsilon, 1]}) = c_{[\epsilon, 1]}bc_{[0, \epsilon]} \neq 0$. Thus, for some $\delta < \epsilon$ sufficiently close to ϵ , we must have $c_{[\epsilon, 1]}bc_{[0, \delta]} \neq 0$ and hence $f(c)bg(c) \neq 0$ where f and g are continuous functions on $[0, 1]$, $f[0, (\epsilon + \delta)/2] = \{0\} = g[(\epsilon + \delta)/2, 1]$ and $f[\epsilon, 1] = \{1\} = g[0, \delta]$. If we had $g(c)bf(b)^2bg(c) \in C^\perp$ then, as $b \in B$, $f(c)bg(c)b = 0$ and hence $f(c)bg(c) = 0$, a contradiction. Thus $f(c)bg(c)a \neq 0$ for some $a \in C$. As C is hereditary and both $f(c)$ and a are in C , this means that $d = f(c)bg(c)a \in C$ and, likewise $d^* \in C$. However, $dd^* \leq \lambda f(c)^2$ for some $\lambda > 0$ while $d^*d \leq \lambda' g(c)^2$ (note that $a, c \in C$ so a commutes with c and hence with $g(c) \in C + \mathbb{C}1$) for some $\lambda' > 0$. As

$f(c)g(c) = 0$, this means that d and d^* do not commute, contradicting the fact C is commutative.

Now the claim is proved, take any $a, b \in B_+$. Given any $c \in C_+$, note that $c(ab - ba) = c^{1/4}ac^{1/2}bc^{1/4} - c^{1/4}bc^{1/2}ac^{1/4} = 0$, as $c^{1/4}ac^{1/4}, c^{1/4}bc^{1/4} \in C$. Thus $ab - ba \in C^\perp \cap B = \{0\}$ and hence, as a and b were arbitrary, B is commutative. \square

10. C*-ALGEBRA TYPE DECOMPOSITIONS

Type classification and decomposition has played a fundamental role in von Neumann algebra theory since its inception almost a century ago. Somewhat analogous type classifications/decompositions have also been obtained for more general C*-algebras, for example in [Cun77] and [CP79]. However, it is only recently that completely consistent extensions of the original von Neumann algebra type decomposition have been obtained by utilizing *-annihilators, either explicitly, as in [Bic14a], or implicitly, as in [NW13].¹¹ In this section, we outline how to obtain order theoretic type decompositions of A and what algebraic characterizations these types have.

First note that, by definition, central elements \mathbb{P}^C in \mathbb{P} lead to finite direct product decompositions. To extend this to infinite products requires separativity. Indeed, we need separativity to first show that the centre of a complete ortholattice \mathbb{P} is a complete sublattice of \mathbb{P} , by [MM70] Corollary (5.14) (although in the case of $\mathcal{P}(A)^\perp$, we know that the centre is $\mathcal{P}(A)^\nabla$, by Theorem 9, which can be easily verified to be a complete sublattice of $\mathcal{P}(A)^\perp$ directly - see [Bic14a]). Then we can define the *central cover* $c(p)$ of $p \in \mathbb{P}$ by

$$c(p) = \bigwedge [p, 1] \cap \mathbb{P}^C.$$

We now get infinite product decompositions as follows.

Theorem 10. *If $(p_\alpha) \subseteq \mathbb{P}$ and $c(p_\alpha) \wedge c(p_\beta) = 0$, for $\alpha \neq \beta$, $[0, \bigvee p_\alpha] \cong \prod [0, p_\alpha]$.*

Proof. See [MM70] Lemma 5.8 and Corollary 5.14. \square

Now say we have some class \mathbf{L} of lattices which is closed under infinite direct products, factors and isomorphisms (which is called a *type class* in [FP10], in the slightly different context of effect algebras). Then Theorem 10 means that,

$$\{p \in \mathbb{P} : [0, p] \in \mathbf{L}\}$$

is \mathbb{P}^C -complete, according to [Bic14b] Definition 2.2. We then get the following type decomposition from [Bic14b] Theorem 2.6 (see also [Bic14b] Theorem 2.4).

Theorem 11. *There exists a unique $p \in \mathbb{P}^C$ such that $p = c(q)$, where $[0, q] \in \mathbf{L}$, and $[0, r] \notin \mathbf{L}$, for all $r \in (0, p^\perp]$.*

¹¹A major stumbling block appears to have been the appropriate extension of the type I concept from von Neumann algebras to C*-algebras. In [CP79] (and earlier in [Gli61]), a C*-algebra is called type I (equivalently, postliminary or GCR) when it has only type I representations. But even type I von Neumann algebras can have non-type I representations, so this does not encapsulate the original meaning of type I. It is rather the concept of a *discrete* C*-algebra, introduced in [PZ00], that consistently extends the notion of a type I von Neumann algebra.

In particular, we can take $\mathbb{P} = \mathcal{P}(A)^\perp$ and, to get decompositions like in the original von Neumann algebra type decompositions (see [MvN36]), we can take \mathbf{L} to be a class exhibiting some degree of distributivity.

To start with, let \mathbf{L} be the class of distributive lattices. By Corollary 7, this decomposition agrees with that given in [Bic14a] Theorem 5.7. Moreover, the $B \in \mathcal{P}(A)^\perp$ corresponding to the p in Theorem 11 is *discrete*, according to [PZ00] Definition 2.1, while B^\perp is *antiliminary*, according to [Ped79] 6.1.1. Thus, this decomposition also agrees with the A_d vs $A_{II} + A_{III}$ part of the decomposition in [NW13] Theorem 5.2. Rephrasing Theorem 11 in this case, we have the following.

Theorem 12. *There exists unique $B, C \in \mathcal{P}(A)^\perp$ with B discrete, C antiliminary and $A = B \vee C$.*

When A is a von Neumann algebra, the B above is the type I part of A , while C is the type II/III part, so this really is completely consistent with the original von Neumann algebra type I vs II/III decomposition. The only key difference between these kinds of decompositions in the von Neumann vs general C*-algebra case is that the supremum \vee here may not correspond to an algebraic direct sum \oplus in general, i.e. we may have $A \neq B \oplus C$, although we do necessarily have $A = (B \oplus C)^{\perp\perp}$, i.e. $B \oplus C$ will be an essential ideal in A .

We can also consider Theorem 11 when $\mathbb{P} = \mathcal{P}(A)^\perp$ and \mathbf{L} is the larger class of *modular* lattices, i.e. satisfying

$$p \leq r \quad \Rightarrow \quad p \vee (q \wedge r) = (p \vee q) \wedge r.$$

Then the $B \in \mathcal{P}(A)^\perp$ corresponding to the p in Theorem 11 is, when A is a von Neumann algebra, precisely the type I/II part of A , while B^\perp is the type III part of A , by the theorem at the start of [Kap55]. In this case it also coincides with the decomposition obtained in [Bic14a] Theorem 6.10 using the relation \sim on *-annihilators (which coincides with Murray-von Neumann equivalence of projections in the von Neumann algebra case), and with the $A_d + A_{II}$ vs A_{III} part of the decomposition obtained in [NW13] Theorem 5.2 using the Cuntz-Pedersen equivalence relation on A_+ . It would seem plausible that the decomposition based on modularity agrees with that based on the \sim relation on *-annihilators even in more general C*-algebras and, indeed, this would follow if a converse to [Bic14a] Theorem 6.14 could be proved. However, UHF algebras are purely infinite with respect to the \sim relation on *-annihilators (see [Bic13a] Proposition 3.89), but finite with respect to the Cuntz-Pedersen equivalence relation on A_+ (as UHF algebras have a faithful trace), so these decompositions do not agree in this case.

We can also consider a slight variant of Theorem 11 when $\mathbb{P} = \mathcal{P}(A)^\perp$ and \mathbf{L} is the class of *orthomodular* lattices, i.e. those ortholattices satisfying

$$p \leq q \quad \Rightarrow \quad p \vee (p^\perp \wedge q) = q.$$

In this case the lattice $[0, q]$ in Theorem 11 must be replaced with the ortholattice $[0, q]^{\perp_q} = \{r^\perp \wedge q : r \leq q\}$ and likewise for $[0, r]$ (and this ortholattice variant of Theorem 11 must be obtained from a similar ortholattice variant of Theorem 10). In the von Neumann algebra case this is not very interesting, as *-annihilators correspond to projections and projections are always orthomodular, i.e. $p = 1$ in this case. But *-annihilators in an arbitrary C*-algebra may not be orthomodular, as Example 4 shows, so p^\perp may be non-zero in this case and one might naturally call this p^\perp the ‘type IV’ part of A . But even though *-annihilators in Example 4 are

not orthomodular, the C^* -algebra in Example 4 still contains a full orthomodular (even commutative) $*$ -annihilator, so it is not type IV (in fact, it is even type I in the restrictive C^* -algebra sense of having only type I representations). Thus this example does not answer the following question.

Question 8. *Do there exist any (non-zero) type IV C^* -algebras? I.e. do there exist C^* -algebras A for which $\mathcal{P}(B)^\perp$ is not orthomodular for any $B \in \mathcal{P}(A)^\perp$?*

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